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A note on α- and α*-Hausdorffness

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Abstract

In this paper, we have introduced α- and α*-Hausdorffness in an *I*-fuzzy topological space (in short, *I*-fts) in the sense of Šostak and Kubiak and characterize them in terms of the corresponding level topologies. It is shown that these notions satisfy the hereditary, productive and projective properties. The degree to which an *I*-fts is α- and α*-Hausdorff, is also discussed.

Keywords: *I*-fuzzy topological spaces; Gradation of openness; Degree of Hausdorffness

1. Introduction

Chang [2] introduced the concept of an *I*-topology on a set *X* as a collection τ of fuzzy sets in *X* which contains $\underline{0}$, $\underline{1}$ and is closed under finite infima and arbitrary suprema. The pair (X, τ) is called an *I*-topological space (in short, *I*-ts), members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. Later it was observed that in this concept, there was no fuzziness involved in the openness or closedness of fuzzy sets. So Šostak [14] and Kubiak [7] introduced an *I*-fuzzy topology on *X* as a map $\tau : I^X \to I$ (*I* being the interval [0, 1]) satisfying the following conditions:

(1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,
(2) $\tau(A_1 \wedge A_2) \geq \min\{\tau(A_1), \tau(A_2)\}$, $A_i \in I^X$, $i = 1, 2$,
(3) $\tau(\bigvee_{i\in\Lambda} A_i) \geq \inf_{i\in\Lambda} \tau(A_i)$, $A_i \in I^X$, $\forall i \in \Lambda$.

For $A \in I^X$, $\tau(A)$ is called the grade of openness of *A* in *X*. The pair (X, τ) is called an *I*-fuzzy topological space (in short, *I*-fts).

Peeters [11] defined α-cuts and α*-cuts of τ in an *I*-fts (X, τ) for $\alpha \in I$ (denoted by $[\tau]_\alpha$ and $[\tau]^*_\alpha$ resp.), as follows:

$$[\tau]_\alpha = \{U \in I^X : \tau(U) \geq \alpha\},$$
$$[\tau]^*_\alpha = \{U \in I^X : \tau(U) > \alpha\}$$

and proved that $\forall\alpha \in I$, $[\tau]_\alpha$ is an *I*-topology on *X*. Here we observed that $[\tau]^*_\alpha$ is a base for an *I*-topology on *X* which we denote by $\phi([\tau]^*_\alpha)$, since $[\tau]^*_\alpha$ is closed under finite intersections and contains *X*. Subspaces and products of *I*-fts′

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are introduced and studied in [11,14]. In [13] Rodabaugh defined α-Hausdorffness (resp. α^*-Hausdorffness) in an I-ts as follows:

An I-ts (X, τ) is called α-Hausdorff (resp. α^* Hausdorff) if $\forall x, y \in X, x \neq y, \exists U, V \in \tau$ such that $U(x) > \alpha$, $V(y) > \alpha$ and $U \wedge V = \underline{0}$ (resp. $U(x) \geq \alpha$, $V(y) \geq \alpha$, $U \wedge V = \underline{0}$).

In this paper we have introduced and studied α- and α^*-Hausdorffness in an I-fts. Several basic results have been obtained which establish the appropriateness of the concept. In particular it is shown that these properties are hereditary, productive and projective. Further it is shown that an I-fts (X, τ) is α^*-Hausdorff iff the diagonal set Δ_X is closed in $(X, [\tau]_\alpha)$ where $\Delta_X = \{(x, x) : x \in X\}$.

2. Preliminaries

Throughout X denotes a set, I denotes the unit interval [0, 1], $I_0 = (0, 1]$, and $I_1 = [0, 1)$. The collection of all fuzzy sets in X is denoted by I^X. The Zadeh complement [19] of an $A \in I^X$ will be denoted by $co A$. The constant fuzzy set in X taking value $\alpha \in I$, is denoted by $\underline{\alpha}$.

Definition 2.1 (*Wong [17]*). A fuzzy point x_r in X is a fuzzy set in X taking value $r \in (0, 1)$ at x and zero elsewhere. A fuzzy singleton [20] x_r in X is a fuzzy set in X taking value $r \in (0, 1]$. x and r are, respectively, called the support and value of x_r. Two fuzzy points/fuzzy singletons are said to be distinct if their supports are distinct. A fuzzy point x_r is said to belong to a fuzzy set A if $r < A(x)$.

It can be easily seen that $x_r \in \bigvee_{i \in \Lambda} A_i \Leftrightarrow x_r \in A_i$ for some $i \in \Lambda$.

Definition 2.2 (*Pu and Liu [9]*). Let x_r be a fuzzy point in X and $A \in I^X$. Then x_r is said to be quasi-coincident with A (notation: $x_r q A$) if $A(x) + r > 1$. Two fuzzy sets A, B in X are said to be quasi-coincident (notation: AqB) if $A(x) + B(x) > 1$ for some $x \in X$. The relation (is not quasi-coincident with) is denoted by $\neg q$. A Q-neighborhood (in short, Q-nbd) of a fuzzy singleton x_r in an I-ts (X, τ) is a fuzzy set $N \in I^X$ such that $\exists U \in \tau$ with $x_r q U \subseteq N$.

Definition 2.3 (*Šostak [14]*). A map $f : (X, \tau) \to (Y, \delta)$ between two I-fts′ is called continuous if $\tau(f^{\leftarrow}(U)) \geq \delta(U)$, $\forall U \in I^X$ where $f^{\leftarrow}$ is defined by $f^{\leftarrow}(U)(x) = U(f(x))$, $\forall x \in X$.

Definition 2.4 (*Peeters [11] and Rodabaugh [13]*). Let (X, τ) be an I-fts and Y be a subset of X. Then $(Y, \tau \mid Y)$ is called a subspace of (X, τ), where $\tau \mid Y : I^X \to I$ is defined by

$$(\tau \mid Y)(U) = \bigvee \{\tau(V) \mid V \in I^X, V \mid Y = U\}.$$

A fuzzy set $A \in I^Y$ is identified with the fuzzy set A' in X such that $A'(x) = A(x)$ if $x \in Y$ and $A'(x) = 0$ if $x \in co Y$ [9].

Now we state the following theorem from Peeters [11].

Theorem 2.1. *Given an arbitrary gradation mapping $t : I^X \longrightarrow I$, let the mappings t^C, t^I and $t^S : I^X \longrightarrow I$ be defined as follows: For all $\mu \in I^X$,*

$$t^C(\mu) = \begin{cases} 1 & \text{if } \mu \text{ is a constant mapping}, \\ t(\mu) & \text{otherwise}, \end{cases}$$

$$t^I(\mu) = \bigvee \left\{ \bigwedge_{i=1}^{n} t(v_i) : n \in N_0, \bigwedge_{i=1}^{n} v_i = \mu \right\},$$

$$t^S(\mu) = \bigvee \left\{ \bigwedge_{i \in j} t(v_i) : \bigvee_{i \in j} v_i = \mu \right\}.$$

Then the mapping $\tau : I^X \longrightarrow I$ where $\tau = t^{CIS} = ((t^C)^I)^S$ is an I-fuzzy topology on X with t as a subbase.