Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 158 (2007) 1873 – 1881
FUZZY sets and systems
www.elsevier.com/locate/fss

Optimal solutions in optimization problem with objective function depending on fuzzy parameters

Jaroslav Ramík*

Department of Mathematical Methods in Economics, Faculty of Business Administration, Silesian University, Karviná, Czech Republic

Available online 19 April 2007

Abstract

In this paper we first introduce a broad class of fuzzy optimization problems with an objective function depending on fuzzy parameters (FO problems) and introduce the concept α-maximal solutions of FO problems with respect to some fuzzy relation. We derive a simple and natural sufficient condition for a feasible solution to be an α-maximal solutions of FO problem. Later on we define possibly and necessarily α-maximal solutions of FO problems and prove that any possibly or necessarily α-maximal solution of FO problem is an α-maximal solution with respect to the possibilistic fuzzy relation or necessity fuzzy relation, respectively. The introduced concepts and results are illustrated and discussed on two numerical examples.

Keywords: Fuzzy optimization; Fuzzy objective function; Fuzzy parameters; Possibly and necessarily optimal solution

1. Introduction

In optimization problems with objective functions having fuzzy parameters, we face the problem whose value is the "best one". Fuzzy numbers, i.e. fuzzy sets of the real numbers, are not linearly ordered, which is why the problem of a suitable concept of optimal solution of these optimization problems, particularly of fuzzy linear programming, has been investigated since the early stage of fuzzy set theory, see e.g. [12].

In this paper we first introduce a broad class of fuzzy optimization problems with an objective function depending on fuzzy parameters (FO problems) and introduce the concept of α-maximal solutions of FO problems with respect to some fuzzy relation. We derive a simple and natural sufficient condition for a feasible solution to be an α-maximal solutions of FO problem. Later we define possibly and necessarily α -maximal solutions of FO problems, see e.g. [12]. We prove that any possibly α-maximal solution of FO problem is an α-maximal solution with respect to the possibilistic fuzzy relation. Analogously, any necessarily α-maximal solution of FO problem is an α-maximal solution with respect to the necessity fuzzy relation. In order to illustrate the introduced concepts and results we present and discuss two numerical examples. Similar approach has been proposed already in several papers by Orlovski, see e.g. [7].

* Tel.: +420 69 6398 227; fax: +420 69 6312 069.
E-mail address: ramik@opf.slu.cz.

0165-0114/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2007.04.003

2. Preliminaries

Let $\mathbf{Z}$ be a nonempty linear topological space. By $F(\mathbf{Z})$ we denote the set of all *fuzzy subsets* A *of* $\mathbf{Z}$, where every fuzzy subset A of $\mathbf{Z}$, or shortly a *fuzzy set*, is uniquely given by the *membership function* $\mu_A : \mathbf{Z} \to [0, 1]$, and $[0, 1] \subset \mathbf{R}$ is a unit interval, $\mathbf{R}$ is the Euclidean space of real numbers. We say that the fuzzy subset A is *crisp* if μ_A is a characteristic function of A, i.e. $\mu_A : \mathbf{Z} \to \{0, 1\}$. It is clear that the set of all subsets of $\mathbf{Z}$ can be isomorphically embedded into $F(\mathbf{Z})$. Let

$$[A]_\alpha = \{x \in \mathbf{Z} | \mu_A(x) \geqslant \alpha\} \quad \text{for } \alpha \in (0, 1],$$
$$[A]_0 = cl\{x \in \mathbf{Z} | \mu_A(x) > 0\},$$

where $cl\,B$ means a topological closure of B, $B \subset Z$. For $\alpha \in [0, 1]$, $[A]_\alpha$ are called α-*cuts*. $[A]_0$ is usually called a *support of* A. By the *strict* α-*cut* we denote

$$(A)_\alpha = \{x \in \mathbf{Z} | \mu_A(x) > \alpha\}.$$

A fuzzy subset A of $\mathbf{Z}$ is *closed, bounded, compact* or *convex*, if $[A]_\alpha$ are closed, bounded, compact or convex subsets of $\mathbf{Z}$ for every $\alpha \in [0, 1]$, respectively. Moreover, A is said to be *normal* if $[A]_1$ is nonempty. It is a well-known fact that a fuzzy subset A of $\mathbf{Z}$ is convex if and only if its membership function μ_A is quasi-concave on $\mathbf{Z}$, see e.g. [11,10]. If A is convex, compact and normal such that $[A]_1$ consists of a single element, then A is called a *fuzzy vector*. For $\mathbf{Z} = \mathbf{R}$ a fuzzy vector is called a *fuzzy number*.

A *binary relation* P *on* $\mathbf{Z}$ is a subset of the Cartesian product $\mathbf{Z} \times \mathbf{Z}$, that is, $P \subset \mathbf{Z} \times \mathbf{Z}$. Here, a *valued relation* P *on* Z is a fuzzy subset of $\mathbf{Z} \times \mathbf{Z}$.

Evidently, any binary relation P on $\mathbf{Z}$ can be isomorphically embedded into the class of valued relations on $\mathbf{Z}$ by its characteristic function (i.e. membership function) μ_P. In this sense, any binary relation is valued.

A fuzzy subset $\tilde{P}$ of $F(\mathbf{Z}) \times F(\mathbf{Z})$ is called a *fuzzy relation on* $\mathbf{Z}$, i.e. $\tilde{P} \in F(F(\mathbf{Z}) \times F(\mathbf{Z}))$.

Let P be a valued relation on $\mathbf{Z}$. A fuzzy relation $\tilde{Q}$ on $\mathbf{Z}$ is called a *fuzzy extension of relation* P, if for each $x, y \in \mathbf{Z}$, it holds

$$\mu_{\tilde{Q}}(x, y) = \mu_P(x, y). \tag{1}$$

Fuzzy relations on $\mathbf{Z}$ are denoted by the tilde, e.g. $\tilde{P}$. $\mathbf{R}^n$ is the n-dimensional Euclidean space, particularly $\mathbf{R}^1 = \mathbf{R}$.

Let A, B be fuzzy sets with the membership functions $\mu_A : \mathbf{R} \to [0, 1]$, $\mu_B : \mathbf{R} \to [0, 1]$, respectively. We consider

$$\mu_{\mathrm{Pos}}(A, B) = \sup\{\min(\mu_A(x), \mu_B(y)) | x \leqslant y, x, y \in \mathbf{R}\}, \tag{2}$$

$$\mu_{\mathrm{Nec}}(A, B) = \inf\{\max(1 - \mu_A(x), 1 - \mu_B(y)) | x > y, x, y \in \mathbf{R}\}. \tag{3}$$

Here (2) is called the *possibility relation on* $\mathbf{R}$, (3) is called the *necessity relation on* $\mathbf{R}$.

The possibility and necessity relations have been originally introduced as *possibility and necessity indices* in [1], where also mathematical analysis and interpretation has been discussed. We write alternatively

$$\mu_{\mathrm{Pos}}(A, B) = (A \preceq^{\mathrm{Pos}} B), \mu_{\mathrm{Nec}}(A, B) = (A \prec^{\mathrm{Nec}} B), \tag{4}$$

where μ_{Pos} and μ_{Nec} are the membership functions of the fuzzy relations on $\mathbf{R}$. By $A \succeq^{\mathrm{Pos}} B$ or $A \succ^{\mathrm{Nec}} B$ we mean $B \preceq^{\mathrm{Pos}} A$ or $B \prec^{\mathrm{Nec}} A$, respectively. In Section 7 we shall deal with possibility and necessity measures.

It can be easily verified that the possibility and necessity relations are fuzzy relations on $\mathbf{R}$, particularly fuzzy extensions of the classical binary relation $\leqslant$, see [9].

3. Optimization problem with fuzzy objective

Let $\mathbf{Y}$ and $\mathbf{Z}$ be topological spaces, $\mathbf{Y}$ is called the *parameter space*, $\mathbf{Z}$ is called the *solution space*. Let X be a given subset of $\mathbf{Z}$, $X \subset \mathbf{Z}$. Consider the following optimization problem with the objective function $f : \mathbf{Y} \times \mathbf{Z} \to \mathbf{R}$ depending on a given *parameter* $C \in \mathbf{Y}$:

$$\begin{array}{ll} \text{maximize} & z = f(C, x) \\ \text{subject to} & x \in X. \end{array} \tag{5}$$