Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 160 (2009) 2017–2031
FUZZY sets and systems
www.elsevier.com/locate/fss

The dominance relation in some families of continuous Archimedean t-norms and copulas

Susanne Saminger-Platz*

Department of Knowledge-Based Mathematical Systems, Johannes Kepler University Linz, A-4040 Linz, Austria

Received 2 September 2008; received in revised form 12 December 2008; accepted 21 December 2008
Available online 30 December 2008

Abstract

The dominance relation in several families of continuous Archimedean t-norms and copulas is investigated. On the one hand, the contribution provides a comprehensive overview on recent conditions and properties of dominance as well as known results for particular cases of families. On the other hand, it contains new results clarifying the dominance relationship in five additional families of continuous Archimedean t-norms and copulas.

Keywords: Fuzzy connectives and aggregation functions; Triangular norms; Copulas; Dominance

1. Introduction

The dominance relation had originally been introduced for triangle functions in the framework of probabilistic metric spaces [47], but was soon abstracted to operations on a partially ordered set [43]. It plays an important role in constructing Cartesian products of probabilistic metric and normed spaces (see [24,43,47], but also [39] for more recent results on dominance between triangle functions resp. operations on distance distribution functions). Dominance, especially between t-norms and copulas, is further crucial in the construction of many-valued equivalence relations [7,8,50] and many-valued order relations 3,4] as well as in the preservation of various properties, most of them expressed by some inequalities, during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [7,11,32,37]. These applications initiated the study of the dominance relation in the broader context of aggregation functions [26,32,37].

Besides these application points of view, dominance has been and is still an interesting mathematical notion. E.g., because of the common neutral element of t-norms and their commutativity and associativity, dominance constitutes a reflexive and antisymmetric relation on the class of all t-norms. Whether the relation is also transitive was of interest already since 1983 (see also [43]). It has been answered recently to the negative by Sarkoci [41] (see also [38]) by means of ordinal sum t-norms meaning that the counter examples have been found in the class of continuous t-norms which form an important subclass of all t-norms.

Obtaining a negative answer has, to some extent, been surprising, since the study of dominance within families of t-norms has been of interest since its very beginning, several particular families of t-norms, containing also subfamilies

* Tel.: +43 702 468 9195; fax: +43 702 468 1351.
E-mail address: susanne.saminger-platz@jku.at
URL: http://www.flll.jku.at.

0165-0114/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2008.12.009

of copulas, had been investigated (see, e.g., [19,33,38,40,44]) and supported the conjecture that the dominance relation would indeed be transitive, either due to its rare occurrence within the family considered or due to its abundant occurrence. Therefore and due to its relevance in applications, it is still of interest to determine whether on some subclasses of t-norms dominance constitutes a transitive and as such an order relation. Particularly interesting are families containing continuous Archimedean t-norms which in its turn most often contain families of Archimedean copulas as subclasses. Many such single-parametric families of t-norms and copulas are listed in Table 2.6 in the book on associative functions by Alsina et al. [2], overlapping to a great extent with the families of Archimedean copulas contained in Table 4.2 in the book on copulas by Nelsen [30].

The aim of the present contribution is to provide results on dominance for several of these families. We pursue this goal in two steps—on the one hand by providing a comprehensive survey on those families for which the dominance relation is already clarified, and on the other hand by proving new results on dominance for five additional families.

Note that, in this contribution, we restrict to dominance among members of a single-parametric family of t-norms. For results comparing members of two different families, see, e.g., [34,36].

The article is organized as follows: In Section 2 some necessary basics on t-norms and Archimedean copulas are summarized. Section 3 contains basic properties and relationships on dominance, in particular dominance among continuous Archimedean t-norms. Section 4 covers the survey on results on dominance known for some of the families contained in [2,30]. Finally, we present new results on dominance for five additional families of t-norms and copulas. We will close the contribution by a short summary.

2. Triangular norms and copulas

We briefly summarize some basic properties of t-norms and copulas for a thorough understanding of this paper (for further details see, e.g., [2,18–22,30,33,37,38,45]).

Definition 1. A *t-norm* $T\colon [0,1]^2 \to [0,1]$ is a binary operation on the unit interval which is commutative, associative, increasing and has neutral element 1.

Well-known examples of t-norms are the *minimum* $T_{\mathbf{M}}$, the *product* $T_{\mathbf{P}}$, the *Łukasiewicz t-norm* $T_{\mathbf{L}}$ and the *drastic product* $T_{\mathbf{D}}$, defined by $T_{\mathbf{M}}(u,v)=\min(u,v)$, $T_{\mathbf{P}}(u,v)=u\cdot v$, $T_{\mathbf{L}}(u,v)=\max(u+v-1,0)$, and

$$T_{\mathbf{D}}(u,v)=\begin{cases}\min(u,v) & \text{if } \max(u,v)=1,\\ 0 & \text{otherwise.}\end{cases}$$

T-norms are compared pointwisely: $T_1 \leqslant T_2$ if $T_1(u,v) \leqslant T_2(u,v)$ for all $u,v\in[0,1]$, expressing that "T_1 is *weaker* than T_2" or "T_2 is *stronger* than T_1". The minimum $T_{\mathbf{M}}$ is the strongest of all t-norms, the drastic product $T_{\mathbf{D}}$ is the weakest of all t-norms.

Definition 2. A t-norm T is called

(i) *Archimedean* if for all $u,v\in\,]0,1[$ there exists an $n\in\mathbb{N}$ such that

$$T(\underbrace{u,\ldots,u}_{n\text{ times}}) < v\,.$$

(ii) A t-norm T is called *strict* if it is continuous and strictly monotone, i.e., for all $u,v,w\in[0,1]$ it holds that

$$T(u,v) < T(u,w) \quad \text{whenever } u>0 \text{ and } v<w\,.$$

(iii) A t-norm T is called *nilpotent* if it is continuous and if each $u\in\,]0,1[$ is a nilpotent element of T, i.e., there exists some $n\in\mathbb{N}$ such that

$$T(\underbrace{u,\ldots,u}_{n\text{ times}}) = 0\,.$$

Of particular interest in the discussion of continuous Archimedean t-norms is the notion of an *additive generator*.