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On (L, M)-fuzzy quasi-uniform spaces

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Abstract

An (L, M)-fuzzy topology is a graded extension of topological spaces handling M-valued families of L-fuzzy subsets of a referential, where L and M are completely distributive lattices. When M reduces to the set $2 = \{0, 1\}$, a $(2, M)$-fuzzy topology is called a fuzzifying topology after Ying. Šostak introduced the notion (L, M)-fuzzy uniform spaces. The aim of this paper is to study the relationship between $(2, M)$-fuzzy quasi-uniform spaces and (L, M)-fuzzy quasi-uniform spaces as well as the relationship between $(2, M)$-fuzzy quasi-uniform spaces and pointwise (L, M)-fuzzy quasi-uniform spaces—the extension of Shi's L-quasi-uniform space in a Kubiak–Šostak sense. It is shown that the category of $(2, M)$-fuzzy quasi-uniform spaces can be embedded in the category of stratified (L, M)-fuzzy quasi-uniform spaces as a both reflective and coreflective full subcategory; and the former category can also be embedded in the category of pointwise (L, M)-fuzzy quasi-uniform spaces.

Keywords: (L, M)-fuzzy topology; (L, M)-fuzzy quasi-uniformity; Pointwise (L, M)-fuzzy quasi-uniformity

1. Introduction

It is well-known that (quasi-)uniformity is a very important concept close to topology and a convenient tool for investigating topology (see [4,13,15]). L-(quasi-)uniformity in Hutton's sense (see [9]) has been accepted by many authors and has attracted wide attention in the literature. Up till now there are many works about the theory of Hutton uniformities (see [6,11,26]). Rodabaugh [16] gave a theory of fuzzy uniformities with applications to the fuzzy real lines. It also needs to point out that Shi [19,20] introduced the theory of pointwise L-quasi-uniformities on fuzzy sets and Shi's theory is simpler and more direct for studying the relationship between pointwise L-quasi-uniformities and pointwise L-topologies.

An (L, M)-fuzzy topology is a graded extension of topological spaces handling M-valued families of L-fuzzy subsets of a referential, where L and M are completely distributive lattices. When M reduces to the set $2 = \{0, 1\}$, a $(2, M)$-fuzzy topology is called a fuzzifying topology after Ying.

Extension of Hutton's quasi-uniformities—[0, 1]-fuzzy uniformity—was considered in [2]. Later, in [22] fuzzy uniformities for lattices more general than [0, 1], namely, the so called (L, M)-fuzzy uniformities were considered. Finally, in [6], a paper specially devoted to the analysis of different approaches to the theory of fuzzy uniformities, an essentially more general concept of an L-valued uniformity was studied using a filter approach. Further, in [18],

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there is a significant extension of Hutton approach for quasi-uniformities without using filters explicitly and without any distributivity and with general tensor products generating the intersection axiom. In [24], the relationship between (L, M)-fuzzy topologies and (L, M)-fuzzy quasi-uniformities was investigated.

Zhang [25] gave a way to embed the category of uniform spaces in the category of Hutton uniform spaces. There is another way to embed the category of uniform spaces in the category of Hutton uniform spaces first in Katsaras [10] for the valued lattice [0, 1] and in Liu and Liang [14] for the general case. One aim of this paper is to study the relationship between $(2, M)$-fuzzy quasi-uniform spaces and (L, M)-fuzzy quasi-uniform spaces by Katsaras and Zhang's approach. It is shown that the category of $(2, M)$-fuzzy quasi-uniform spaces can be embedded in the category of stratified (L, M)-fuzzy quasi-uniform spaces as a both reflective and coreflective full subcategory. Another aim of this paper is to study the relationship between $(2, M)$-fuzzy quasi-uniform spaces and pointwise (L, M)-fuzzy quasi-uniform spaces—the extension of Shi's L-quasi-uniform space in a Kubiak–Šostak sense. It is shown that the category of $(2, M)$-fuzzy quasi-uniform spaces can be embedded in the category of pointwise (L, M)-fuzzy quasi-uniform spaces.

2. Preliminaries

Let L be a complete lattice. An element $a \in L$ is said to be coprime (resp., prime) if $a \leqslant b \vee c$ (resp., $a \geqslant b \wedge c$) implies that $a \leqslant b$ or $a \leqslant c$ (resp., $a \geqslant b$ or $a \geqslant c$). The set of all coprimes (primes) of L is denoted by $c(L)$(resp., $p(L)$). We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$) or $b \gg a$ ($b \rhd a$), if for every directed (arbitrary) subset $D \subseteq L$, $\bigvee D \geqslant b$ implies $a \leqslant d$ for some $d \in D$. Clearly if $a \in L$ is a coprime, then $a \ll b$ if and only if $a \lhd b$. A complete lattice L is said to be continuous (completely distributive) if every element in L is the supremum of all the elements way below (wedge below) it.

Proposition 2.1 (*Gierz et al. [5]*). *Let L be a complete lattice. The following conditions are equivalent*:

(1) *L is completely distributive*;
(2) *L is distributive continuous lattice with enough coprimes*;
(3) *The operator $\bigvee : Low(L) \to L$ sending every lower set to its supremum has a left adjoint β, and in this case $\beta(a) = \{b \mid b \lhd a\}$.*

From (3) in the above proposition, it is easy to see that the wedge below relation has the interpolation property in a completely distributive lattice, this is to say, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$. For more detail about completely distributive lattices, please refer to [5].

In the following, L and M are two completely distributive lattices and L possesses an order reversing involution $'$. L^X is the set of all L-fuzzy sets on X. $A' \in L^X$ is defined by $A'(x) = (A(x))'$. The set of all coprimes of L^X is denoted by $c(L^X)$. χ_U denotes the characteristic function of $U \in 2^X$. Let $F : X \to Y$ be an ordinary mapping, define $F_L^{\to} : L^X \to L^Y$ and $F_L^{\leftarrow} : L^Y \to L^X$ by $F_L^{\to}(A)(y) = \bigvee\{A(x) \mid x \in X, F(x) = y\}$ for $A \in L^X$ and $y \in Y$, and $F_L^{\leftarrow}(B)(x) = B(F(x))$ for $B \in L^Y$ and $x \in X$ (following the notation in [17]), respectively.

Definition 2.2 (*Höhle [7], Höhle and Šostak [8], Kubiak [12], Šostak [21]*). An (L, M)-fuzzy topology is a mapping $\tau : L^X \to M$ such that

(FCT1) $\tau(1_X) = \tau(0_X) = 1$;
(FCT2) $\tau(A \wedge B) \geqslant \tau(A) \wedge \tau(B)$ for all $A, B \in L^X$;
(FCT3) $\tau(\bigvee_{j \in J} A_j) \geqslant \bigwedge_{j \in J} \tau(A_j)$ for every family $\{A_j \mid j \in J\} \subseteq L^X$.

The pair (L^X, τ) is called an (L, M)-fuzzy topological space. A mapping $F : (L^X, \tau) \to (L^Y, \tau_1)$ is said to be continuous with respect to τ and τ_1 if $\tau(F_L^{\leftarrow}(B)) \geqslant \tau_1(B)$ for all $B \in L^Y$. Let (L, M)-**FTOP** denote the category of (L, M)-fuzzy topological spaces and continuous mappings.

When $L = \{0, 1\}$, Definition 2.2 will reduce to that of M-fuzzifying topology. Let M-**FYS** denote the category of M-fuzzifying topological spaces.