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A unifying study between modal-like operators, topologies and fuzzy sets

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Abstract

The paper presents the essential connections between modal-like operators, topologies and fuzzy sets. We show, for example, that each fuzzy set determines a preorder and an Alexandrov topology, and that similar correspondences hold also for the other direction. Further, a category for preorder-based fuzzy sets is defined, and it is shown that its equivalent subcategory of representatives is isomorphic to the categories of preordered sets and Alexandrov spaces. Moreover, joins, meets and complements for the objects in this category of representatives are determined. This suggests how to define for fuzzy subsets of a certain universe the lattice operations in a canonical way.

Keywords: Modal-like operators; Preorders; Fuzzy sets; Topologies; Category theory

1. Background

The current paper studies relationships between modal-like operators defined by means of preorders, topologies and fuzzy sets, where the term fuzzy sets includes the more general L-sets in which L may be any preordered set. The technical report [17] should be considered as a preliminary version of the current paper, although many enhancements in presentation and notation are incorporated here, as well as some new ideas.

Topologies closed under arbitrary intersections play an essential role in the current work, and here they are called as *Alexandrov topologies*, because it seems to be the most used name in contemporary literary. It should be noted that in the literature Alexandrov topologies are referred to by various names: Alexandrov [3] introduced these topologies under the name *discrete topologies*, Birkhoff [5] named them as *completely distributive topologies*, and Steiner [32] studied so-called *principal topologies* and proved that a topology is principal if and only if it is closed under arbitrary intersections, thus, an Alexandrov topology. Also other terms, such as *smallest-neighbourhood spaces* can be found in the literature, but they are not recalled here.

Birkhoff [5] showed that there exists an one-to-one correspondence between Alexandrov topologies and preorders. Additionally, Steiner [32] showed that when ordered by the set-inclusion relation, the sets of all preorders and Alexandrov topologies on U are dually order-isomorphic. Further, these ordered sets are complemented complete lattices

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(but not usually distributive). It is also a well-known result that the categories of preordered sets and Alexandrov spaces are isomorphic, as discussed in [2], for example.

Fuzzy sets can be considered simply as functions from a universe to a certain preordered set. Kortelainen [20–22] has introduced an approach to study fuzzy sets by presenting that for each fuzzy set we can attach a preorder $\lesssim$ such that $x \lesssim y$, if x belongs to the set in question at most at the same extent as y. This then means that every fuzzy set determines a preorder—and an Alexandrov topology. Further, Kripke [23] has given semantics for necessity and possibility operators in modal logic by means of relational operators. Therefore, it is quite natural to consider modal operators determined by the preorder $\lesssim$ and its inverse $\gtrsim$; these operators are here called *modal-like operators*. Interestingly, in the Alexandrov space induced by a fuzzy set, the smallest neighbourhood, interior, and closure of any set can be expressed simply in terms of modal-like operators.

The paper is organized as follows: In Section 2 we give well-known preliminaries for modal-like operators. Section 3 recalls essential connections between Alexandrov spaces and preordered sets. Then, in Section 4 we recall L-sets and present relationships between modal-like operators defined by means of a preorder, Alexandrov topologies and L-sets. Finally, in Section 5 we give a categorical interpretation for the essential relationships between these structures.

2. Modal-like operators

As mentioned in the previous section, by modal-like operators we mean relation-based operators syntactically similar to classical modal operators. Let R be an arbitrary binary relation on U and let us denote for all $x \in U$, $R(x) = \{y \in U \mid x\,R\,y\}$. We define for any $X \subseteq U$,

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\} \quad \text{and} \quad X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}.$$

It is well-known that the operators ${}^{\blacktriangle}\colon \wp(U) \to \wp(U)$ and ${}^{\blacktriangledown}\colon \wp(U) \to \wp(U)$ are *dual*, that is, for all $X \subseteq U$,

$$X^{c\blacktriangle} = X^{\blacktriangledown c} \quad \text{and} \quad X^{c\blacktriangledown} = X^{\blacktriangle c},$$

where X^c is the *complement* $U \setminus X$ of X. Further, the operator ${}^{\blacktriangle}$ is completely union-preserving and the operator ${}^{\blacktriangledown}$ is completely intersection-preserving, that is, $\bigcup \mathcal{H}^{\blacktriangle} = \left(\bigcup \mathcal{H}\right)^{\blacktriangle}$ and $\bigcap \mathcal{H}^{\blacktriangledown} = (\cap \mathcal{H})^{\blacktriangledown}$ for all $\mathcal{H} \subseteq \wp(U)$; see [18]. The above implies easily that $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are dually order-isomorphic complete lattices; the isomorphism is $X^{\blacktriangledown} \mapsto X^{c\blacktriangle}$.

Of course, we can define another pair of operators on $\wp(U)$ by means of the *inverse* $R^{-1} = \{(x, y) \mid y\,R\,x\}$ of R. Indeed, for any $X \subseteq U$,

$$X^{\vartriangle} = \left\{x \in U \mid R^{-1}(x) \cap X \neq \emptyset\right\} \quad \text{and} \quad X^{\triangledown} = \left\{x \in U \mid R^{-1}(x) \subseteq X\right\}.$$

The operators ${}^{\blacktriangle}$, ${}^{\blacktriangledown}$, ${}^{\vartriangle}$, ${}^{\triangledown}$ are called here *modal-like operators*. The operators ${}^{\blacktriangle}$ and ${}^{\blacktriangledown}$ may be viewed as *rough approximation operators* [13–15,28], and the operators ${}^{\vartriangle}$ and ${}^{\triangledown}$ are sometimes interpreted as *modifier operators* [20,25]. Also other types of definitions for these operators can be found in the literature; see [7] for a survey. Additionally, the sets $R(x)$ and $R^{-1}(x)$ can be considered as neighbourhoods of $x \in U$ and relational interpretations of *neighbourhood operators* were considered in [33].

In the following, we present some interesting connections between the operators ${}^{\blacktriangle}$, ${}^{\blacktriangledown}$, ${}^{\vartriangle}$, and ${}^{\triangledown}$. For two ordered sets P and Q, a pair $({}^{\blacktriangleright}, {}^{\blacktriangleleft})$ of mappings ${}^{\blacktriangleright}\colon P \to Q$ and ${}^{\blacktriangleleft}\colon Q \to P$ is a *Galois connection between P and Q* if for all $p \in P$ and $q \in Q$,

$$p^{\blacktriangleright} \leqslant q \iff p \leqslant q^{\blacktriangleleft}.$$

It is well-known that the pairs $({}^{\vartriangle}, {}^{\blacktriangledown})$ and $({}^{\blacktriangle}, {}^{\triangledown})$ form Galois connections on $\wp(U)$; see, for example [9,27]. This implies, for example, that the mappings $X \mapsto X^{\vartriangle\blacktriangledown}$ and $X \mapsto X^{\blacktriangle\triangledown}$ are lattice-theoretical closure operators, and $X \mapsto X^{\triangledown\blacktriangle}$ and $X \mapsto X^{\blacktriangledown\vartriangle}$ are lattice-theoretical interior operators. Furthermore,

$$X^{\blacktriangle} = X^{\blacktriangle\triangledown\blacktriangle}, \quad X^{\triangledown} = X^{\triangledown\blacktriangle\triangledown}, \quad X^{\vartriangle} = X^{\vartriangle\blacktriangledown\vartriangle}, \quad X^{\blacktriangledown} = X^{\blacktriangledown\vartriangle\blacktriangledown}$$