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Statistical limit inferior and limit superior for sequences of fuzzy numbers

Salih Aytar[a,*], Musa A. Mammadov[b], Serpil Pehlivan[a]

[a] *Suleyman Demirel University, Department of Mathematics, Isparta, Turkey*
[b] *University of Ballarat, Ballarat, Australia*

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Abstract

In this paper, we extend the concepts of statistical limit superior and limit inferior (as introduced by Fridy and Orhan [Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125 (12) (1997) 3625–3631. [12]]) to statistically bounded sequences of fuzzy numbers and give some fuzzy-analogues of properties of statistical limit superior and limit inferior for sequences of real numbers.

Keywords: Fuzzy number; Statistical convergence; Statistical boundedness; Statistical limit superior and limit inferior

1. Introduction and background

The idea of statistical convergence was introduced by Steinhaus [26] and also independently by Fast [8] and Buck [3] for real and complex sequences. Salat [24] used the idea of bounded statistical convergence to construct the sequence space which is a nowhere dense subset of the normed linear space l_∞ of all bounded sequences of real numbers. On the other hand, Maddox [16] extended the concept for sequences in any Hausdorff locally convex topological vector spaces. In the case of real sequences, Fridy [10] obtained the statistical analogue of Cauchy criterion of convergence. These concepts were used in Turnpike theory as an application [17,22]. Recently, Teran [27] showed a general method to translate Tauberian theorems for summability methods in $\mathbb{R}$ into Tauberian theorems for the corresponding forms of statistical convergence in metric spaces.

Bounded and convergent sequences of fuzzy numbers were first introduced by Matloka [18] that every convergent sequence is bounded. In [19], Nanda studied the spaces of bounded and convergent sequences of fuzzy numbers and proved that these are complete metric spaces. On the other hand, Nuray and Savaş [21] introduced and discussed the concept of statistically convergent and statistically Cauchy sequences of fuzzy numbers. Later, Nuray [20], Savaş [25] and Kwon [14,15] offered fuzzy-analogues of some results in the theory of statistical convergence. Aytar [1] has recently examined the concepts of statistical limit and cluster point for sequences of fuzzy numbers.

So far "sup" and "inf" notions have been given only for bounded sets of fuzzy numbers (see [7,28]), and in this study we introduce the notions of statistical limit superior and limit inferior for statistically bounded sequences of fuzzy

* Corresponding author. Tel.: +90 246 211 4120; fax: +90 246 237 1106.
E-mail addresses: aytar@fef.sdu.edu.tr (S. Aytar), m.mammadov@ballarat.edu.au (M.A. Mammadov), serpil@sdu.edu.tr (S. Pehlivan).

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numbers. We then prove that some results established for sequences of real numbers in [12] are also valid for sequences of fuzzy numbers. But the proposition: "If $st\text{-}\liminf_{k\to\infty} x_k = st\text{-}\limsup_{k\to\infty} x_k = x_0$ then $st\text{-}\lim_{k\to\infty} x_k = x_0$", which is important for sequences of real numbers $x = (x_k)$, may not be valid for sequences of fuzzy numbers, as it can be seen in Example 2 below. In this paper we give a version of this proposition which is true for any statistically bounded sequence of fuzzy numbers.

Briefly, we recall some of the basic notations in the theory of fuzzy numbers and we refer readers to [4–6,13,18,28] for more details.

Given an interval A, we denote its endpoints by $\underline{A}$ and $\overline{A}$. We denote the set of all closed intervals on the real line $\mathbb{R}$ by D. That is,

$$D := \{A \subset \mathbb{R} : A = [\underline{A}, \overline{A}]\}.$$

For $A, B \in D$ define

$$A \leqslant B \text{ iff } \underline{A} \leqslant \underline{B} \text{ and } \overline{A} \leqslant \overline{B},$$
$$d(A, B) := \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|).$$

It is easy to see that d defines a metric (Hausdorff metric) on D and (D, d) is a complete metric space. Also '$\leqslant$' is a partial order in D.

A fuzzy number is a function X from $\mathbb{R}$ to [0,1], satisfying

- X is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;
- X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $X(\lambda x + (1-\lambda)y) \geqslant \min\{X(x), X(y)\}$;
- X is upper semi-continuous;
- the closure of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact.

These properties imply that for each $\alpha \in (0, 1]$, the α-level set $X^\alpha := \{x \in \mathbb{R} : X(x) \geqslant \alpha\} = [\underline{X}^\alpha, \overline{X}^\alpha]$ is a nonempty compact convex subset of $\mathbb{R}$, as the support X^0: $X^0 = \lim_{\alpha\to 0^+} X^\alpha$. We denote the set of all fuzzy numbers by $L(\mathbb{R})$.

Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \longrightarrow \mathbb{R}$ by

$$\overline{d}(X, Y) := \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha).$$

Puri and Ralescu [23] showed that $L(\mathbb{R})$ is a complete metric space with the metric $\overline{d}$.

For $X, Y \in L(\mathbb{R})$ define

$$X \leqslant Y \text{ iff } X^\alpha \leqslant Y^\alpha \text{ for any } \alpha \in [0, 1].$$

We say that $X < Y$ if $X \leqslant Y$ and there exists $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$.

The fuzzy numbers X and Y are said to be incomparable if neither $X \leqslant Y$ nor $Y \leqslant X$. We use the notation $X \not\sim Y$ in this case.

A subset E of $L(\mathbb{R})$ is said to be bounded from above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leqslant \mu$ for every $X \in E$. μ is called the least upper bound(sup) of E if μ is an upper bound and $\mu \leqslant \mu'$ for all upper bounds μ'. A lower bound and the greatest lower bound (inf) are defined dually. E is said to be bounded if it is both bounded from above and bounded from below. In addition, Wu and Wu [28] prove that if the set $E \subset L(\mathbb{R})$ is bounded then its supremum and infimum exist (see also [7]).

For every $X, Y, Z \in L(\mathbb{R})$, we say that Z is the sum of X and Y, written $Z = X + Y$, if for every $\alpha \in [0, 1]$, $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$.

Consider a fuzzy number $\mu \in L(\mathbb{R})$. Let $\mu^\alpha := [\underline{\mu}^\alpha, \overline{\mu}^\alpha]$, $\alpha \in [0, 1]$, be α-level sets of μ. Given a positive number $a > 0$, we define the fuzzy numbers $\mu + a_1$ and $\mu - a_1$ as follows [13]:

$$(\mu + a_1)^\alpha := [\underline{\mu}^\alpha, \overline{\mu}^\alpha] + [a, a] = [\underline{\mu}^\alpha + a, \overline{\mu}^\alpha + a],$$

$$(\mu - a_1)^\alpha := [\underline{\mu}^\alpha, \overline{\mu}^\alpha] - [a, a] = [\underline{\mu}^\alpha - a, \overline{\mu}^\alpha - a],$$

where

$$a_1(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$