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Fuzzy Sets and Systems 157 (2006) 719 – 738

FUZZY sets and systems

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L-uniform spaces versus $\mathbb{I}(L)$-uniform spaces[☆]

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Received 2 September 2004; received in revised form 11 October 2005; accepted 11 October 2005
Available online 7 November 2005

Abstract

This paper studies relationships between the categories of $\mathbb{I}(L)$-uniform spaces, L-uniform spaces and uniform spaces. We construct two adjunctions: $\Phi_L \dashv \Psi_L$ between the category of $\mathbb{I}(L)$-uniform spaces and the category of L-uniform spaces and $\phi_L \dashv \psi_L$ between the category of L-uniform spaces and the category of uniform spaces (with L a complete lattice with an order-reversing involution in both cases), which with $L = \{0, 1\}$ and $L = \mathbb{I} = [0, 1]$, respectively, reduce to the adjunction from the category of $\mathbb{I}$-uniform spaces to the category of uniform spaces investigated by Katsaras. If L is any complete lattice (not necessarily with an order-reversing involution), then so is the L-unit interval $\mathbb{I}(L)$, and we have another adjunction, viz. $\phi_{\mathbb{I}(L)} \dashv \psi_{\mathbb{I}(L)}$ from the category of $\mathbb{I}(L)$-uniform spaces to the category of uniform spaces. We show that the following two factorizations hold: $\phi_{\mathbb{I}(L)} = \Phi_L \circ \phi_L$ and $\psi_{\mathbb{I}(L)} = \psi_L \circ \Psi_L$. When L is a meet-continuous lattice with an order-reversing involution, there is also a natural link between $\Phi_L \dashv \Psi_L$ and the existing adjunction $\Omega_L \dashv \mathrm{I}_L$ from the category of $\mathbb{I}(L)$-topological spaces to the one of L-topological spaces, via the forgetful functors. An essential tool in this paper is the theory of Galois connections. It is emphasized that no kind of distributivity is assumed on the lattice L in this paper.

Keywords: Uniformity; L-uniformity; Unit L-interval; $\mathbb{I}(L)$-uniformity; Adjoint functors; L-topology; $\mathbb{I}(L)$-topology; Galois connection

1. Introduction

In the literature there are two essentially different generalizations of the classical Lowen adjunction $\omega_{\mathbb{I}} \dashv \iota_{\mathbb{I}}$ from the category **TOP**($\mathbb{I}$) of $\mathbb{I}$-topological spaces to **TOP** [21] (where $\mathbb{I} = [0, 1]$). The first generalization, $\omega_L \dashv \iota_L$, replaces the unit interval $\mathbb{I}$ by an arbitrary continuous lattice L (cf. [15,27,16]), while the second one replaces the unit interval $\mathbb{I}$ by the Hutton's unit L-interval $\mathbb{I}(L)$ with L a meet-continuous lattice (cf. [12–14,18]) and provides an adjunction $\Omega_L \dashv \mathrm{I}_L$ from **TOP**($\mathbb{I}(L)$) to **TOP**(L) which with $L = \{0, 1\}$ reduces to $\omega_{\mathbb{I}} \dashv \iota_{\mathbb{I}}$ (cf. [13,14]). These two different approaches were recently shown to be related through the factorizations: $\Omega_L \circ \omega_L = \omega_{\mathbb{I}(L)}$ and $\iota_L \circ \mathrm{I}_L = \iota_{\mathbb{I}(L)}$ with L a hypercontinuous lattice endowed with an order-reversing involution (cf. [19]).

It seems natural to ask to what extent one can have a similar situation between the category **UNIF**(L) of L-uniform spaces and L-uniformly continuous maps (in the sense of Hutton [8]) and the classical category **UNIF**. It was Katsaras [9] who with $L = \mathbb{I} = [0, 1]$ developed an adjunction $\phi_{\mathbb{I}} \dashv \psi_{\mathbb{I}}$ from **UNIF**($\mathbb{I}$) to **UNIF** and showed that there is a

☆ Partially supported by UPV-00127.310-E-15429/2003.
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doi:10.1016/j.fss.2005.10.012

natural link between the functors $\omega_{\mathbb{I}}$, $\phi_{\mathbb{I}}$, and the forgetful functors which replace the ($\mathbb{I}$-)uniformity by the ($\mathbb{I}$-)topology generated by it, that is, he proved that if $\mathcal{U}$ is a uniformity, then the $\mathbb{I}$-topology generated by the $\mathbb{I}$-uniformity $\phi_{\mathbb{I}}(\mathcal{U})$ coincides with the image of the topology generated by $\mathcal{U}$ under the functor $\omega_{\mathbb{I}}$.

The aim of this paper is to study relationships between the three categories **UNIF**($\mathbb{I}(L)$), **UNIF**(L), and **UNIF**. We construct two essentially different generalizations of the adjunction $\phi_{\mathbb{I}} \dashv \psi_{\mathbb{I}}$ of Katsaras, the first one, denoted by $\phi_L \dashv \psi_L$, between **UNIF**(L) and **UNIF** and the second one, which we will denote by $\Phi_L \dashv \Psi_L$, between **UNIF**($\mathbb{I}(L)$) and **UNIF**(L), being L a complete lattice with an order-reversing involution in both cases (note that $\mathbb{I}(L)$ is complete whenever L is so and when L has an order-reversing involution, then $\mathbb{I}(L)$ has the induced order-reversing involution). We note that ϕ_L with L a completely distributive lattice is already present in [3,20]. An essential tool in these constructions is the theory of Galois connections. We then show that these two approaches of generalizing the adjunction $\phi_{\mathbb{I}} \dashv \psi_{\mathbb{I}}$ are related in the same way as in the context of L-topologies since the functors $\phi_{\mathbb{I}(L)}$ and $\psi_{\mathbb{I}(L)}$ will be shown to have the factorizations $\phi_{\mathbb{I}(L)} = \Phi_L \circ \phi_L$ and $\psi_{\mathbb{I}(L)} = \psi_L \circ \Psi_L$. We finally show that, when L is a meet-continuous lattice with an order-reversing involution, there is also a natural link between the functors Ω_L and Φ_L since $\Omega_L \circ \mathrm{T}_L$ coincides with $\mathrm{T}_{\mathbb{I}(L)} \circ \Phi_L$ where T_L is the forgetful functor from **UNIF**(L) to **TOP**(L).

We note that our relationships between the categories **UNIF**(L) and **UNIF**($\mathbb{I}(L)$) can be viewed as a particular case of external change of basis (cf. [6]).

We emphasize that this paper is distributivity-free. All the material related to L-uniformities is developed in a complete lattice setting (with the existence of an order-reversing involution). One specific assumption, exploited in Section 5, about the underlying lattice L is the meet-continuity. It is needed to assert that, given an L-uniform space $(X, \mathcal{D})$, the family $T_{\mathcal{D}} = \{a \in L^X : a = \bigvee\{b \in L^X : \exists D \in \mathcal{D}, D(b) \leqslant a\}\}$ is an L-topology on X. This has previously been known to hold for L a frame (= an infinitely distributive lattice); cf. [26, Proposition 6.2]. Also, meet-continuity is required when speaking about L-topologically generated $\mathbb{I}(L)$-topological spaces.

2. Preliminaries

2.1. Lattices and L-topological spaces

In this paper $L = (L, \leqslant)$ denotes a complete lattice (with bounds denoted $\perp$ and $\top$). The two-point lattice $\{\perp, \top\}$ will be denoted by **2**. Given $\alpha \in L$, we let $\uparrow \alpha = \{\beta \in L : \alpha \leqslant \beta\}$ and dually for $\downarrow \alpha$.

When L is endowed with an order-reversing involution $'$, i.e., given $\alpha, \beta \in L$ one has $\alpha'' = \alpha$ and $\beta' \leqslant \alpha'$ whenever $\alpha \leqslant \beta$, then we write $(L, ')$.

The only specific assumptions about L, occasionally used in this paper, will be *meet-continuity* (cf. [1]):

$$\alpha \wedge \bigvee D = \bigvee\{\alpha \wedge \gamma : \gamma \in D\}$$

for each $\alpha \in L$ and every directed subset $D \subset L$.

Convention. A careful distinction is made between L and $(L, ')$. We shall refer just to L if it is not assumed that the lattice L admits an order-reversing involution. Saying that $(L, ')$ is complete (or meet-continuous) is supposed to mean that the lattice $L = (L, \leqslant)$ is assumed to be complete (or meet-continuous) and, moreover, endowed with $'$.

We also recall a number of standard L-topological concepts. Given a set X, L^X denotes the complete lattice of all maps from X to L (called L-sets) ordered pointwisely; whenever we have $(L, ')$, L^X has an order-reversing involution defined by $a'(x) = a(x)'$ $(a \in L^X,\ x \in X)$. If $A \subset X$, then $1_A \in L^X$ denotes the characteristic function of A. The constant member of L^X with value α is denoted α too. Given a map $f : X \to Y$, $a \in L^X$ and $b \in L^Y$, we define the usual (Zadeh) image and preimage operators:

$$f^{\to}(a) = \bigvee_{x \in X} a(x) \wedge 1_{\{f(x)\}}$$

and

$$f^{\leftarrow}(b) = b \circ f,$$