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Non-exchangeable copulas and multivariate total positivity

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ABSTRACT

Multivariate total positivity of order 2 (MTP2) is a dependence property with a number of applications in statistics and mathematics. Given the theoretical and practical relevance of MTP2, it is important to investigate the conditions under which random vectors have this property. In this paper we contribute to the development of the theory of stochastic dependence by employing the general concept of copula. In particular, we propose a new family of non-exchangeable Archimedean copulas which leads to MTP2. The focus on non-exchangeability allows us to overcome the limitations induced by symmetric dependence, typical of standard Archimedean copulas.

1. Introduction

Total positivity is a concept of considerable interest in many fields of statistics and mathematics. In particular, multivariate total posectsitivity of order 2 (MTP2, see [1]) has a number of applications in statistical decision procedures, multivariate analysis, simultaneous statistical inference, approximating probabilities, and reliability theory. The MTP2 property is known to be satisfied by a fairly limited number of multivariate distributions (see e.g. [1,2]) and the concept of copula can be very useful in extending the family of known MTP2 random vectors. In this vein [3] derive necessary and sufficient conditions for the generator of an Archimedean copula to yield a random vector which is MTP2. It is important to note that Archimedean copulas (see e.g. [4–7] for some recent results and extensive surveys), are among the most relevant examples of exchangeable copulas. Exchangeability is a very important property, satisfied by a qualified family of distributions. However, despite its mathematical relevance, exchangeability may represent a requirement too strong to be commonly fulfilled by a set of random variables. Therefore, it is worth paying attention to the concept of non-exchangeable generalization of Archimedean copulas (see [8,9]).

In this paper, we move from [8,9] and provide a new family of asymmetric copulas generated by a one-dimensional function which leads to MTP2 (see Theorem 3.3). In doing this, we extend [3] to the case of non-exchangeability for the considered vector of dependent random variables. Therefore, using Theorem 3.3 we can provide sufficient conditions for a vector of non-symmetrically dependent random variables to be MTP2: note also that the random variables are not required to possess a special joint distribution and can even have different marginals. The theoretical result leads to the identification of a new family of copulas associated to the MTP2 property (see Proposition 3.4).

The rest of the paper is organized as follows. Section 2 provides the necessary preliminaries, notation and statistical concepts. The main results are offered in Section 3. Some concluding remarks are given in Section 4.

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2. Preliminaries and notation

For the sake of simplicity, we introduce the vectorial notation:

Notation 2.1. *Fix* $m = 1, 2, \ldots$. *The following notations are introduced:* $\boldsymbol{W} = (w_1, \ldots, w_m)$ *is a random vector;* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m)$ *are elements of* $\mathbb{R}^m$ *and* $\boldsymbol{u} = (u_1, \ldots, u_m) \in [0, 1]^m$.

We now recall the definition of the dependence concept we deal with.

Definition 2.2. Let f be the joint density function of the m-variate random vector $\boldsymbol{W}$. The components of $\boldsymbol{W}$ are said to be MTP2 if and only if, for each $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^m$, it results:

$$f(\boldsymbol{x}) \cdot f(\boldsymbol{y}) \leq f(\min\{\boldsymbol{x}, \boldsymbol{y}\}) \cdot f(\max\{\boldsymbol{x}, \boldsymbol{y}\})$$

where the min and max operators are meant component-wise.

Definition 2.2 formalizes a (not necessarily linear) dependence structure of positive type. When dealing with (linear) dependence among individual w's in $\boldsymbol{W}$, it is customary to consider a non-diagonal variance-covariance matrix $\Sigma = (\sigma_{i,j})$ with $i, j = 1, \ldots, m$. Hence, it is natural to guess the existence of a relationship between the value (and the sign) of the covariances and the validity of MTP2. In this respect, it is worth recalling a standard result which states that if $\{w_1, \ldots, w_m\}$ are MTP2, then $\sigma_{i,j} \geq 0$, for each $i, j = 1, \ldots, m$ (see e.g. [1]). This fact implies that if there exists a couple (w_i, w_j), with $i \neq j$ and $i, j = 1, \ldots, m$, such that $\sigma_{i,j} < 0$, then $\{w_1, \ldots, w_m\}$ are not MTP2.

A rather general way to capture the stochastic dependence structure among random variables is the introduction of the concept of *multivariate copula* or, simply, *copula* (we refer to [10] for a detailed discussion). In particular, Sklar's Theorem [11] highlights how multivariate copulas model the dependence structure among random variables (see e.g. [10, Section 2.3]).

A popular family of copulas that found a number of applications is the Archimedean one, and [3] derive the conditions for an Archimedean copula to give rise to a MTP2 random vector. For the reader's convenience we recall here the definition of Archimedean copula:

Definition 2.3. An Archimedean copula is a function C: $[0, 1]^m \to [0, 1]$ of the form

$$C(\boldsymbol{u}) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_m)) \quad \text{for } u_i \in [0, 1]$$

where the copula generator function $\varphi : [0, 1] \to [0, +\infty]$ is a strictly decreasing function with $\lim_{t \to 0+} \varphi(t) = \infty$, $\varphi(1) = 0$, and φ^{-1} is d-monotonic.

An advantage of Archimedean copulas, that partly explains their theoretical and empirical success, is that they can represent a wide range of dependence properties, according to the specific generator function φ. An important feature of some classes of copulas, including the Archimedean one, is *exchangeability* (see e.g. [12]).

We recall here the definition of exchangeable copulas.

Definition 2.4. The copula C: $[0, 1]^m \to [0, 1]$ is exchangeable if, for each $\boldsymbol{u} \in [0, 1]^m$ and for each permutation ϱ of $\{1, \ldots, m\}$, one has:

$$C(\boldsymbol{u}) = C(u_{\varrho(1)}, \ldots, u_{\varrho(m)}).$$

When Definition 2.4 is not satisfied, then copula C is said to be non-exchangeable.

Exchangeability implies symmetric dependence, which is typically modeled with Archimedean copulas by using just one or two parameters. This can be an undesirable property in practice. For this reason the development of new non-exchangeable copulas and the study of their properties are important fields of theoretical research.

3. Main result

The family of non-exchangeable copulas we deal with is generated by a one-dimensional function, and represents a generalization of the usual Archimedean copulas. We formalize it in the following:

Definition 3.1. Fix $J \in \mathbb{N}$ and a set of $m \times J$ functions

$$h_{jk} : [0, 1] \to [0, 1], \qquad j = 1, \ldots, J; \; k = 1, \ldots, m \tag{1}$$

such that:

(C3.1.i) h_{jk} is differentiable in (0, 1) and strictly increasing in [0, 1], for all j, k;
(C3.1.ii) $h_{jk}(0) = 0$ and $h_{jk}(1) = 1$, for all j, k;
(C3.1.iii) $\frac{1}{J} \sum_{j=1}^{J} h_{jk}(x) = x$, for each $k = 1, \ldots, m$ and $x \in [0, 1]$. Moreover, define

$$\psi : [0, 1] \to [0, 1] \tag{2}$$

such that:

(C3.1.iv) ψ is $m + 2$ times differentiable in (0, 1);