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Multivariate upper semilinear copulas

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ABSTRACT

In the problem setting of constructing an n-copula given its diagonal section and all of its $(n-1)$-dimensional marginals, we introduce a new class of symmetric n-copulas, which generalizes the well-known class of bivariate upper semilinear copulas. These new upper semilinear n-copulas are constructed by linear interpolation on segments connecting the main diagonal of the unit hypercube $[0, 1]^n$ to one of its upper faces. We focus on the case where the $(n-1)$-dimensional marginals are upper semilinear $(n-1)$-copulas themselves, in which case the n-copula is actually constructed given its diagonal section and the diagonal sections of its k-marginals ($k \in \{2, 3, \dots, n-1\}$). We provide necessary and sufficient conditions on these diagonal sections that guarantee that the upper semilinear construction method yields an n-copula. Several examples are provided.

1. Introduction

Copulas were first introduced in a statistical context through Sklar's theorem [27], which states that the joint cumulative distribution function of a vector of n continuous random variables can be uniquely represented by its corresponding marginal distribution functions and an n-copula. For example, the product copula $\Pi_n(\mathbf{x}) = x_1x_2\dots x_n$ is associated to a vector of independent random variables, while the comonotonicity copula $M_n(\mathbf{x}) = \min(x_1, x_2, \dots, x_n)$ is associated to a vector of random variables that are perfectly positive dependent. An n-copula can also be interpreted as the joint cumulative distributive function of an n-dimensional random vector whose univariate marginal distributions are uniformly distributed on the interval [0, 1]. While most of the studies focus on the bivariate case, recently several studies have explored the higher-dimensional setting [1,2,6,8,14,24].

Let us explicitly recall that an n-copula C is a $[0, 1]^n \to [0, 1]$ function that satisfies the following conditions:

(c1) $C(\mathbf{x}) = 0$ if $\mathbf{x}$ is such that $x_j = 0$ for some $j \in \{1, 2, \dots, n\}$.
(c2) $C(\mathbf{x}) = x_j$ if $\mathbf{x}$ is such that $x_i = 1$ for all $i \neq j$ and some $j \in \{1, 2, \dots, n\}$.
(c3) C is n-increasing, i.e., for any n-box $\mathbf{P} = \times_{j=1}^{n} [a_j, b_j] \subseteq [0, 1]^n$, it holds that:

$$V_C(\mathbf{P}) = \sum_{\mathbf{x}\in \text{vertices}(\mathbf{P})} (-1)^{S(\mathbf{x})} C(\mathbf{x}) \geq 0,$$

where $S(\mathbf{x}) = \#\{j \in \{1, 2, \dots, n\} \mid x_j = a_j\}$. $V_C(P)$ is called the C-volume of P.

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When $n-k$ of the arguments of an n-copula are set equal to 1, we obtain a k-marginal of the copula, which itself is a k-copula.

The diagonal section d_n of an n-copula C is the $[0, 1] \to [0, 1]$ function defined by $d_n(x) = C(x, x, \ldots, x)$. It is easy to check that this function has the following properties:

(d1) d_n is increasing.
(d2) $d_n(1) = 1$.
(d3) $d_n(x) \leq x$.
(d4) d_n is n-Lipschitz continuous, i.e.,

$$|d_n(y) - d_n(x)| \leq n|y - x|.$$

for any $x, y \in [0, 1]$.

Note that condition (d3) implies that $d_n(0) = 0$. A function that satisfies (d1)–(d4), for some integer n, is called an *n-diagonal function*. From a probabilistic point of view, the diagonal section of an n-copula is the distribution function of the maximum order statistic of uniform random variables on [0, 1] that have the given n-copula as joint distribution function. Note that, by definition, any n-diagonal function is also a k-diagonal function for any $k \geq n$. It can be shown that for any n-diagonal function d_n, there exists an n-copula that has d_n as a diagonal section (see [3,14]).

In the bivariate case, several methods are available to construct copulas with a given diagonal section. Some examples are:

(i) The diagonal copula [26], given by:

$$D_{d_2}(x_1, x_2) = \min\left(x_1, x_2, \frac{d_2(x) + d_2(y)}{2}\right),$$

is the greatest symmetric 2-copula with given diagonal section d_2.

(ii) The Bertino copula [12], given by:

$$B_{d_2}(x_1, x_2) = \min(x_1, x_2) + \sup\{d_2(t) - t \mid t \in [\min(x_1, x_2), \max(x_1, x_2)]\},$$

is the smallest 2-copula with given diagonal section d_2.

There exist several other ways of constructing bivariate copulas with a given diagonal section, such as the (upper and lower) semilinear construction method, which is the main source of inspiration for the present work. Semilinear copulas were first introduced by Durante et al. [7]. These copulas are constructed by linearly interpolating between the values at the lower boundaries (condition (c1)) or upper boundaries (condition (c2)) of the unit square and the values at the diagonal given by the 2-diagonal function d_2. Several generalizations of this approach have been proposed, for example, construction methods that linearly interpolate on other segments of the unit square [4,5,16–18,20,21] or construction methods that use a polynomial interpolation [11,19,22]; see also [10] for other generalizations. Recently there has been a growing interest to see whether the above-mentioned construction methods that were developed for the bivariate case can be generalized to the n-dimensional case. For instance, the concept of diagonal copula in n dimensions has been studied in [14], whereas the present authors have investigated the generalization of the Bertino copula in n dimensions [2].

The aim of this paper is to study the possible generalization of the upper semilinear 2-copulas to the n-dimensional case. This paper is organized as follows. In the next section, we recall the definition of upper and lower semilinear 2-copulas and present a construction method for the multivariate case. In Section 3, we characterize the corresponding set of diagonal sections. In Section 4, we provide some examples.

2. Construction method

Semilinear copulas were first introduced by Durante et al. [7]. A 2-copula C is called upper semilinear if the mappings

$$h_1 : [t_0, 1] \to [0, 1], \quad h_1(x) = C(x, t_0)$$

$$v_1 : [t_0, 1] \to [0, 1], \quad v_1(x) = C(t_0, x)$$

are linear for all $t_0 \in [0, 1]$. It has been proven that the bivariate function U_{d_2}, defined by

$$U_{d_2}(x_1, x_2) = \frac{(x_{(2)} - x_{(1)})x_{(1)} + (1 - x_{(2)})d_2(x_{(1)})}{1 - x_{(1)}}, \tag{1}$$

where $x_{(1)} = \min(x_1, x_2)$ and $x_{(2)} = \max(x_1, x_2)$ and the convention $0/0 = 1$ is adopted, is the upper semilinear 2-copula with diagonal section d_2, provided that the following conditions hold:

(i) The function $\nu_{d_2} : [0, 1[, \to [0, \infty[$, defined by $\nu_{d_2}(x) = (x - d_2(x))/(1 - x)$, is increasing.
(ii) The function $\phi_{d_2} : [0, 1[, \to [0, \infty[$, defined by $\phi_{d_2}(x) = (1 - 2x + d_2(x))/(1 - x)^2$, is increasing.