Information Sciences 317 (2015) 259–277

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Type-2 triangular norms and their residual operators

Li Dechao *

School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316022, China
Key Laboratory of Oceanographic Big Data Mining and Application of Zhejiang Province, Zhoushan 316022, China

A R T I C L E   I N F O

Article history:
Received 8 November 2014
Received in revised form 28 April 2015
Accepted 2 May 2015
Available online 7 May 2015

Keywords:
Type-2 fuzzy sets
Type-2 *t*-norms
Residuation
Compositional rule of inference

A B S T R A C T

In this paper, we mainly investigate *T*-extension operations of any *t*-norms and *t*-conorms on type-2 fuzzy sets' truth values $\mathscr{F}_2$ (a set of all functions defined from $[0,1]$ into itself). Based on it, we first construct some type-2 *t*-norms on the fuzzy truth values $\mathscr{F}_2$ with the ordinary partial order $\leqslant$ and the partial order $\sqsubseteq$, respectively. The algebraic properties of these type-2 *t*-norms are then studied. Moreover, the residual operators of some special type-2 *t*-norms on $(\mathscr{F}_2, \leqslant)$ and $(\mathscr{F}_2, \sqsubseteq)$ are respectively represented. Finally, we briefly discuss the compositional rule of inference based on type-2 *t*-norms and their residual operators.

1. Introduction

Although classical (type-1) fuzzy sets became the "language" of vague propositions, their $[0,1]$-valued truth values are still precise. In order to strengthen the capability of modeling and manipulating inexact information in a logical manner, the concept of type-2 fuzzy sets was introduced by Zadeh [33], extending the notion of type-1 fuzzy sets. Type-2 fuzzy sets assign a fuzzy set to each element on its domain. Since type-2 fuzzy sets own more parameters than type-1 fuzzy sets, they can provide us with more design degrees of freedom in practice. Moreover, type-2 fuzzy sets seem to provide a better framework for the "computing with words" paradigm than classical ones [14,23]. In recent years, type-2 fuzzy sets became increasingly important in many aspects [3–5,13,22,24,30]. Especially, type-2 fuzzy sets had been successfully employed in different control applications [2,9,11,15,20,21,35]. It is worth mentioned that the computational complexity of type-2 fuzzy set operations is the main constraint on application.

It becomes more meaningful to consider what the types of operators are possible for type-2 fuzzy sets in practical applications. Type-2 *t*-norm is an indispensable tool modeling the intersection of type-2 fuzzy sets. As a result, it becomes a favorite topic to systematically study of type-2 *t*-norms from the mathematical point of view. It is well known that some type-2 *t*-norms can be derived directly from Zadeh' extension principle. Many researchers have been studied extended *t*-norms in accord ance with Zadeh' extension principle. For example, Mizumoto and Tanaka firstly studied the set-theoretic operations and the algebraic structures of type-2 fuzzy sets under the operations $\wedge_T$, where *T* is one of the four basic *t*-norms on $[0,1]$ [25,26]. Karnik and Mendel provided some general formulas for the extended *t*-norms and *t*-conorms on finite type-2 fuzzy sets which have discrete domains [16]. Gera and Dombi represented some computationally pointwise formulas for extended *t*-norms and *t*-conorms on type-2 fuzzy sets [7]. Starczewski gave expressions of some extended *t*-norms for fuzzy truth intervals or fuzzy truth numbers [28]. Hu and Kwong discussed properties of *t*-norm extension operations of general binary operation for fuzzy true values on a linearly ordered set, with a unit interval and a real number set as special cases [12].

* Address: School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316022, China.
E-mail address: dch1831@163.com

http://dx.doi.org/10.1016/j.ins.2015.05.003

Classical t-norm is a binary operation defined on the linearly ordered set $[0,1]$. As an extension of classical t-norm, type-2 t-norm is defined on fuzzy truth values of type-2 fuzzy sets, which is the set of all functions from $[0,1]$ into itself. Due to the close connection between order sets and fuzzy sets, we can study type-2 t-norms from the view of partial order sets. Unlike $[0,1]$, the fuzzy truth values of type-2 fuzzy sets can be equipped with two different partial orders $\leqslant$ and $\sqsubseteq$, respectively [31]. And then they become two complete lattices, respectively. This triggers us to treat type-2 t-norms as triangular norms on a general partial order set. Moreover, a left continuous t-norm plays an essential role in fuzzy set theory. Therefore, we mainly investigate the continuity of extended t-norms in accordance with Zadeh's extension principle and their residual operators on the partial order sets $(\mathscr{F}_2, \leqslant)$ and $(\mathscr{F}_2, \sqsubseteq)$ in this paper. Having this in mind, this paper is organized as follows. In Section 2, we give some definitions of basic notions and notations. Section 3 constructs some type-2 t-norms, their residual operators and t-conorms on the partial order set $(\mathscr{F}_2, \leqslant)$. In Section 4, some algebraic properties of type-2 t-norms on the partial order sets $(\mathscr{F}_2, \sqsubseteq)$ are investigated, and then the residual operators of some special type-2 t-norms on $(\mathscr{F}_2, \sqsubseteq)$ are shown. Section 5 discusses the compositional rule of inference based on left continuous type-2 t-norms and their residual operators.

2. Preliminaries

First, we briefly summarize some basic concepts and results that are needed for further treatment. Let X and Y be two universes of discourse and $Map(X,Y)$ denotes the set of all mappings from X to Y. A type-1 fuzzy set A can be regarded as an element of $Map(X,[0,1])$. The family of all fuzzy sets on X is denoted by $\mathscr{F}(X)$. Common operations on $\mathscr{F}(X)$ are $\cap$ (intersection), $\cup$ (union), and c (complement) given pointwise by

$$\begin{aligned} A\cap B(x) &= \min\{A(x),B(x)\},\\ A\cup B(x) &= \max\{A(x),B(x)\},\\ A^c(x) &= (A(x))', \end{aligned} \tag{1}$$

and the two nullary operations are given by $\emptyset(x)=0$ and $X(x)=1$ for all $x\in X$. It is well known that $(\mathscr{F}(X),\cap,\cup,c,\emptyset,X)$ is a bounded distributive lattice with an involutive c that satisfies De Morgan's laws, that is, it is De Morgan algebra. However, it should be noted that min, max and c are not the only one type of operation for complement, union, or intersection. More general, t-norms, t-conorms and negation are used to be qualified as these operations [16].

Let J be a linearly ordered set with an involutive negation N. If for any $A\subseteq J$ it holds $\inf A\in J$ and $\sup A\in J$, then J is called complete. If J is bounded, then the smallest and greatest elements in J are written as 0 and 1, respectively. In this case, the algebra is written as $(J,\vee,\wedge,N,0,1)$, which was discussed in [31]. In this paper, we consider the case which $(J,\vee,\wedge,N,0,1)$ is De Morgan algebra, that is, it is a bounded distributive complete lattice satisfying the De Morgan laws.

Definition 2.1 (*[25,26]*). A type-2 fuzzy set $\widetilde{A}$ in a universe X is characterized by a fuzzy membership function A as

$$\widetilde{A}: X\to Map(J,[0,1]) \tag{2}$$

with the value $\widetilde{A}(x)$ being called a fuzzy grade. A fuzzy grade $\widetilde{A}(x)$ can be represented by

$$\widetilde{A}(x) = f_x(u), \tag{3}$$

where f_x is a membership function for the fuzzy grade $\widetilde{A}(x)$ which is defined as $f_x: J\to[0,1]$.

The family of all type-2 fuzzy sets on X is denoted by $\mathscr{F}_2(X)$. The algebraic operations in $\mathscr{F}_2(X)$ are determined from Zadeh's principle of extension.

Definition 2.2 (*[25,26]*). Let $\widetilde{A},\widetilde{B}\in\mathscr{F}_2(X)$, i.e., $\widetilde{A}(x)=\{f_x|x\in X, f_x: J\to[0,1]\}$ and $\widetilde{B}(x)=\{g_x|x\in X, g_x: J\to[0,1]\}$. The membership grades for union ($\sqcup$), intersection ($\sqcap$) and complement ($\neg$) of $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

$$\begin{aligned} (\widetilde{A}\sqcup\widetilde{B})(x) &= \bigvee_{x=u\vee v}(f(u)\wedge g(v)),\\ (\widetilde{A}\sqcap\widetilde{B})(x) &= \bigvee_{x=u\wedge v}(f(u)\wedge g(v)),\\ (\neg\widetilde{A})(x) &= \bigvee_{x=u'} f(u) = f(x'),\\ \widetilde{0}(x) &= \begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases},\quad \widetilde{1}(x) = \begin{cases}1 & x=1\\ 0 & x\neq 1\end{cases}, \end{aligned} \tag{4}$$

where $\vee,\wedge$ and $'$ are maximum, minimum and involutive negation on [0, 1], respectively.

Definition 2.3 [31]. Let $\widetilde{A},\widetilde{B}\in\mathscr{F}_2(X)$. Two partial orders are defined as follows: