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Solution method for a boundary value problem with fuzzy forcing function

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ABSTRACT

In this paper, we present a new approach to a non-homogeneous fuzzy boundary value problem. We consider a linear differential equation with real coefficients but with a fuzzy forcing function and fuzzy boundary values. We assume that the forcing function is a triangular fuzzy function. Unlike previous studies, we look for a solution that is a fuzzy set of real functions (not a fuzzy-valued function). Each of these real functions satisfies the boundary value problem with some membership degree. We have developed a method that finds this solution, and demonstrated its effectiveness using a test example.

To show that the approach can be extended to other types of fuzzy numbers, we extended it to the trapezoidal case. For a particular example, we used the product t-norm to demonstrate how a new solution type can be obtained.

1. Introduction

1.1. Related work

Many researchers have studied fuzzy differential equations, and, in particular, fuzzy boundary value problems (FBVPs) [1–6,8–38,40–51,53–55,59–67].

FBVPs are important because they arise in many applications. Zarei et al. [69] recently proposed a fuzzy mathematical model of HIV infection, which consists of a linear fuzzy differential equation. The authors considered a fuzzy optimal control problem, minimizing both the viral load and the drug costs. They obtained an optimality condition in the form of an FBVP. Jafelice et al. [35] proposed a model for the evolution of the positive HIV population and manifestation of AIDS. The authors considered the transference rate of HIV to AIDS as a fuzzy set. This transference rate depends on the viral load and the CD4+ level of infected individuals. Salahshour and Haghi [66] solved the fuzzy heat equation under strongly generalized H-differentiability. Using the fuzzy Laplace transform, they converted the original fuzzy heat equation into the corresponding fuzzy two-point boundary value problem.

FBVPs were first considered by Lakshmikantham et al. [46] and by Saito [64]. These researchers and O'Regan et al. [62] stated that a two-point boundary value problem for a fuzzy differential equation is equivalent to a fuzzy integral equation. However, Bede [10] used a counterexample to disprove this. He also proved that two-point boundary value problems have no solutions in many cases. Nevertheless, under a new structure and certain conditions, Chen et al. [20,21] showed that a

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two-point boundary value problem is equivalent to a fuzzy integral equation. They also proved the existence of the solution to the problem. Saito [65] proved the existence and uniqueness theorem for the solution of an FBVP by applying Schauder's fixed point theorem and the contraction principle. Agarval et al. [1] studied the existence of fuzzy solutions for multipoint boundary value problems. Arara and Benchohra [8] used Banach's fixed point theorem to establish the existence and uniqueness of fuzzy solutions for a class of second-order, nonlinear boundary value problems with integral boundary conditions.

Murty and Kumar [51] developed existence and uniqueness criteria for a certain class of boundary value problems for third-order nonlinear fuzzy differential equations, with the help of Green's functions and the contraction mapping principle. Prakash et al. [61] also presented a three-point boundary value problem using Green's functions, but for another type of boundary conditions.

Khastan and Nieto [42] interpreted a two-point boundary value problem for a second-order fuzzy differential equation using generalized differentiability, and proposed a new solution concept. Li et al. [48] introduced the concept of big solutions to two-point boundary value problems of undamped, uncertain dynamical systems. They proved the existence and uniqueness of big solutions. Nieto et al. [55] proved the existence and uniqueness of solution for a first-order linear fuzzy differential equation with impulses subject to boundary value conditions. Liu [49] investigated the support of solutions for two-point fuzzy boundary value problems. Fard et al. [24] presented some sufficient conditions for the existence and uniqueness of a solution, using Hukuhara differentiability. Rodríguez-López [63] provided sufficient conditions for the existence of solutions to periodic boundary value problems for first-order linear fuzzy differential equations, using generalized differentiability and switching points.

Khastan et al. [44] investigated the existence of solutions for a class of FBVPs under generalized differentiability. Nieto and Rodríguez-López [54] calculated the exact solution for a class of FBVPs for first-order fuzzy linear differential equations with impulses under Hukuhara differentiability. Ahmadi et al. [3] solved a fuzzy second-order differential equation using the Laplace transform on the fuzzy, strongly generalized derivative. Prakash et al. [60] used a differential transformation method to find the solution of second-order two-point and third-order three-point FBVPs. Esfahani et al. [23] proved the existence and uniqueness theorem for FBVPs of first-order, fuzzy, nonlinear equations under generalized differentiability. Allahviranloo and Chehlabi [5] considered fuzzy differential equations based on the concept of the length function.

There are many publications regarding numerical methods for FVBPs. Allahviranloo and Khalilpour [6] proposed the finite-difference method for the numerical solution of an FBVP. Khalilpour and Allahviranloo [40] presented an initial-value method for solving FBVPs. Fatullayev and Köroglu [25] proposed an algorithm that used the finite difference method to numerically solve FBVPs. Bede and Rudas [12] proposed a shooting algorithm for numerically solving fuzzy, two-point boundary value problems such as a fuzzy elastica problem. Jamshidi and Avazpour [34] applied the shooting method to solve second order, fuzzy boundary value differential equations under generalized differentiability. Dahalan et al. [22] studied numerical solutions to FBVPs using the Gauss–Seidel and successive over relaxation iterative methods.

Approaches to FBVPs depend on the concept of the solution to the fuzzy differential equation. There are two main approaches. In the first, because the solution is a fuzzy function, researchers assume that the derivative in the differential equation is a fuzzy derivative. This derivative can be the Hukuhara derivative, or a derivative in the generalized sense. Bede [10] demonstrated that a large class of boundary value problems do not have a solution when using the Hukukara derivative. To avoid this, Bede and Gal [11] developed the concept of the generalized derivative. Khastan et al. [41,45] and Khastan and Nieto [42] investigated fuzzy differential equations using this concept. In [2], Ahmad et al. studied analytical and numerical solutions of fuzzy differential equations based on the extension principle. By considering the dependency problem in fuzzy interval arithmetic, the authors proposed a new fuzzification of Euler's method. Recently, Akın et al. [4] presented a new algorithm for solving fuzzy differential equations with the generalized derivative. They first solve the corresponding classical problem, and then construct the fuzzy solution by assuming that it is "close" to the classical solution. Using novel generalizations of the Hukuhara difference for fuzzy sets, Bede and Stefanini [14] investigated new generalized differentiability concepts for fuzzy-valued functions.

The second approach is based on generating the fuzzy solution from the classical solution. When considering the fuzzy initial value problem, this approach can be realized in three ways. The first method uses the extension principle. We solve the associated classical problem, and then replace the real initial value in the solution with the fuzzy one. In the final solution, arithmetic operations are considered to be operations on fuzzy numbers (Buckley and Feuring [15,16]). The second method was proposed by Hüllermeier [33] and uses the concept of differential inclusion. This method takes an α-cut of the initial value and the solution function, converting the differential equation into a differential inclusion. The obtained solution is accepted as the α-cut of the fuzzy solution. Misukoshi et al. [50] proved that, under certain conditions, these two main approaches are equivalent for the initial value problem. In the third method, the solution is considered to be a fuzzy set of real functions. This method was proposed by Gasilov et al. [26,29], Barros et al. [9], and Gomes and Barros [32]. Barros et al. [9], and Gomes and Barros [32] introduced the concepts of fuzzy calculus (analogous to classical calculus) and applied it to solve fuzzy differential equations. Gasilov et al. [26,27] considered the properties of linear transformations and proposed a method for finding a fuzzy bunch of solution functions for a linear equation.

1.2. Motivation

The behavior of a dynamical system is described mainly by a differential equation. In a real-world problem, some parameters in the equation are determined from measurements (or, observations) and may contain uncertainties. Often these