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Permutable fuzzy consequence and interior operators and their connection with fuzzy relations

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ABSTRACT

Fuzzy operators are an essential tool in many fields and the operation of composition is often needed. In general, composition is not a commutative operation. However, it is very useful to have operators for which the order of composition does not affect the result. In this paper, we analyze when permutability appears. That is, when the order of application of the operators does not change the outcome. We characterize permutability in the case of the composition of fuzzy consequence operators and the dual case of fuzzy interior operators. We prove that for these cases, permutability is completely connected to the preservation of the operator type.

We also study the particular case of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule and the inf-$\rightarrow$ composition. For this cases, we connect permutability of the fuzzy relations (using the sup-$*$ composition) with permutability of the induced operators. Special attention is paid to the cases of operators induced by fuzzy preorders and similarities. Finally, we use these results to relate the operator induced by the transitive closure of the composition of two reflexive fuzzy relations with the closure of the operator this composition induces.

1. Introduction

Fuzzy consequence operators and fuzzy interior operators are an essential tool in most of the different frameworks were fuzzy logic appears. As prominent examples, we find approximate reasoning and fuzzy mathematical morphology. In approximate reasoning, fuzzy consequence operators are used to obtain conclusions from certain fuzzy premises and fuzzy relations [13,18,19,30]. Fuzzy interior operators appear as a dual notion of fuzzy consequence operators in the lattice of truth values [5]. In fuzzy mathematical morphology, fuzzy consequence operators and fuzzy interior operators are called fuzzy closings and openings respectively and they act as morphological filters used for image processing [7,8,15,16]. Operators induced by fuzzy relations appear in this context as a generalization of morphological filters defined in sets were an additive operation does not necessarily exist [20,22]. In these cases, the fuzzy relation plays the role of structuring element. This abstraction allows to use certain techniques from fuzzy mathematical morphology into data mining problems [21]. Other places where fuzzy consequence and interior operators appear are modal logic [10], fuzzy topology [23–25], fuzzy rough sets [9,28,34], fuzzy relation equations [29] and fuzzy concept analysis [1,3].

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In all these contexts there is a need of concatenating two or more operators and it is important to know when this composition preserves their properties. A very relevant question is whether the composition of two fuzzy consequence (interior) operators is such an operator. As will be shown it this paper, it turns out to be closely related to their permutability or commutativity.

The objective of this paper is to characterize permutability in the case of composition of either two fuzzy consequence operators or two fuzzy interior operators. We study two particular cases of operators induced by fuzzy relations: fuzzy operators induced by means of Zadeh compositional rule and fuzzy operators induced by the inf-$\rightarrow$ composition, mainly focusing on the cases of operators induced by fuzzy preorders and similarities. As we shall see, permutability of fuzzy relations is closely related to permutability of their induced fuzzy operators and preservation of their properties.

The paper is organized as follows. In Section 2 we set the framework and we recall the main definitions and results that will be used throughout the paper.

In Section 3 we recollect several definitions and results that show connections between fuzzy relations and fuzzy operators. We recall the operators C^*_R and $C^{\rightarrow}_R$ given by Zadeh's compositional rule and inf-$\rightarrow$ composition respectively, several of their properties and extend the notion of C^*_R to a more general process to obtain fuzzy operators from a fuzzy relation and another fuzzy operator. We introduce the notion of concordance between a fuzzy operator and a fuzzy relation, which is the key to preserve the properties of fuzzy consequence operator of the induced operator.

Sections 4 and 5 are devoted to the analysis of permutability for certain cases of fuzzy relations and fuzzy operators. In Section 4, we study permutability of general fuzzy preorders and the particular case of fuzzy indistinguishability relations. In Section 5, permutability of fuzzy consequence operators is characterized and dual results are obtained for the case of fuzzy interior operators.

Sections 6 and 7 show the relationship between permutability of fuzzy relations and permutability of fuzzy operators by using the connections established in Section 3. In Section 6, we relate permutability of fuzzy relations with permutability of the operators that they induce through Zadeh's compositional rule. In Section 7, a similar study is made for operators induced through inf-$\rightarrow$ composition. We use the results developed in Sections 4 and 5 in order to study the cases of fuzzy operators induced by fuzzy preorders and similarities.

In Section 8 we analyze under which conditions different properties of the induced operators are satisfied even if permutability does not hold. Some of these properties are used to relate the operator induced by the transitive closure of the composition of two reflexive fuzzy relations with the closure of the operator this composition induces.

Finally, in Section 9 we present the conclusions.

2. Preliminaries

Let X be a non-empty classical set and let $[0,1]^X$ denote the set of all fuzzy subsets of X with truth values in $[0,1]$ endowed with the structure of complete commutative residuated lattice (in the sense of Bělohlávek [4]). That is, $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ where $\wedge$ and $\vee$ are the usual infimum and supremum, $*$ is a left-continuous t-norm and $\rightarrow$ is the residuum of $*$ defined for $\forall a, b \in X$ as $a \rightarrow b = \sup\{\gamma \in [0,1] | a * \gamma \leqslant b\}$.

Recall that $*$ and $\rightarrow$ satisfy the adjoiness property

$$x * y \leqslant z \Longleftrightarrow y \leqslant x \rightarrow z$$

and that $*$ is monotone in both arguments while $\rightarrow$ is antitone in the first argument and monotone in the second one.

As always, the inclusion of fuzzy sets is defined by the pointwise order, i.e. $\mu \subseteq \nu$ if and only if $\mu(x) \leqslant \nu(x)$ for all $x \in X$.

Let us recall some properties of $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ that will be used in the paper. Detailed proofs can be found in [4].

Proposition 2.1. *The residuated lattice* $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ *satisfies the following conditions for each index set I and for all* $x, x_i, y, y_i, z \in [0,1]$ *with* $i \in I$*:*

1. $1 \rightarrow x = x$	6. $(x \rightarrow y) * (y \rightarrow z) \leqslant (x \rightarrow z)$
2. $x \leqslant y \Longleftrightarrow x \rightarrow y = 1$	7. $x * \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x * y_i)$
3. $x * 0 = 0$	8. $x \rightarrow \bigwedge_{i\in I} y_i = \bigwedge_{i\in I}(x \rightarrow y_i)$
4. $x * (x \rightarrow y) \leqslant y$	9. $\bigvee_{i\in I} x_i \rightarrow y = \bigwedge_{i\in I}(x_i \rightarrow y)$
5. $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$	$x * \bigwedge_{i\in I} y_i \leqslant \bigwedge_{i\in I}(x * y_i)$

We will use the notation sup or $\vee$ for the supremum and inf or $\wedge$ for the infimum indistinctly.

Recall that every partially ordered set P, and therefore every lattice, gives rise to a dual (or opposite) partially ordered set which usually denoted P^δ. P^δ is defined to be the set P with the inverse order, i.e. $x \leqslant y$ holds in P^δ if and only if $y \leqslant x$ holds in P. It is easy to see that this construction allows us to translate every statement from P to a statement P^δ by replacing each occurrence of $\leqslant$ by $\geqslant$. Notice that if P is a lattice, every occurrence of $\vee$ gets replaced by $\wedge$ and vice versa [14].