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Combining various types of belief structures

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ABSTRACT

We first discuss the basic ideas of the Dempster–Shafer belief structure. We particularly emphasize the associated measures of plausibility and belief. We discuss the cumulative distribution function associated with a belief structure. We next turn to the issue of combining multiple belief structures. Two different types of combination of belief structures are investigated. The first, which we refer to as the fusion of belief structures, occurs when the belief structures being combined are providing information about the same variable. Here we make use of the Dempster rule. The second type, which we refer to as the joining of belief structures, occurs when the belief structures being combined are providing information about different variables. Here we make use of a belief structures cumulative distribution function and the Sklar theorem. The classic belief structures typically have a finite number of focal elements; here we began to look at belief structures which have a continuum of focal elements.

1. Introduction

The Dempster–Shafer provides a framework for representing uncertain information about the value of a variable [1–8]. It is particularly applicable in situations in which the information about the uncertainty contains aspects of both randomness and granular imprecision. A particular important task in use of belief structures is the combining of information from multiple sources. Here we consider two types of combination of belief structures. The first, which we refer to as fusion, occurs when the information being combined are about the same variable. Here the result is some belief structure about the variable of interest. The second type, which we refer to as joining, occurs when the information being combined are about the different variables. Here we obtain a joint belief structure. In this case use is made of the aggregation via copulas of the associated cumulative distribution functions [9,10]. In the final part of this work we look to extend the representational capability of the D-S belief structure by allowing for belief structures in which we can have a continuum of focal elements. Among other things this allows us to model imprecise probability density functions. We show how to combine these types of continuum based belief structures.

2. Dempster–Shafer belief structures

Formally a classic Dempster–Shafer belief structure, m, is defined on a space X and consists of a finite collection of non-empty subsets $B_1, \ldots, B_n$ of X called focal elements and a mapping m: $\{B_1, \ldots, B_n\} \rightarrow [0, 1]$ so that m $(B_i) > 0$ and $\sum_{j=1}^{n} m(B_j) = 1$ [3]. The space X can be discrete or a continuous subset of the real line. While the Dempster–Shafer structure

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is not a fuzzy measure, as it is not monotonic, a number of fuzzy measures can be associated with it. Two important ones are the measures of plausibility, Pl and belief, Bel. We recall [3]

$$Pl(A) = \sum_{B_j \cap A \neq \emptyset} m(B_j)$$
$$Bel(A) = \sum_{B_j \subseteq A} m(B_j)$$

Another measure associated with a D-S belief structure m is the pignistic measure of Smets [11,12],

$$Pig(A) = \sum_{j=1}^{n} \frac{Card(B_j \cap A)}{Card(B_j)} m(B_j)$$

It is easy to show that the pignistic measure is a probability measure: if $E \cap F = \emptyset$ then $Pig(E \cup F) = Pig(E) + Pig(F)$.

It is well known that $Bel(A) \leqslant Pig(A) \leqslant Pl(A)$ and that $Pl(X) = Bel(X) = 1$ and $Pl(\emptyset) = Bel(\emptyset) = 0$. It is also known that these measures are monotonic.

One important application of the Dempster–Shafter belief structure is to provide information about an uncertain variable V whose domain is X. We shall use the notation V is m to denote this situation.

A number of different semantics have been associated with the D-S belief structure [7,13]. For our purposes we shall use the one initially introduced by Dempster [2]. We shall let V be a random variable taking its value in the space X. Associated with the random variable V is some unknown probability measure P. The D-S structure provides a framework for representing knowledge about this underlying probability measure in the case in which we do not have precise knowledge of the values associated with the probability measure P. This has come to be called type-two uncertainty. In classic probability theory for each element in X we have an associated probability. In our situation the D-S structure is interpreted so that $m(B_i)$ is an amount of probability allocate to the elements in B_i in an unknown manner. Thus here, instead of allocating the total probability mass of one directly to the elements in X we associate this probability mass to the subsets B_i. Thus $m(B_i)$ is an amount of probability shared among the elements in B_i.

Under this interpretation of the D-S structure, for any subset A of X, the plausibility $Pl(A)$, provides an upper bound on the probability of A and the belief, $Bel(A)$, provides a lower bound that probability of A. Thus $Bel(A) \leqslant Prob(A) \leqslant Pl(A)$. Here then our knowledge of $Prob(A)$ is that it lies in a range, $Prob(A) \in [Bel(A), Pl(A)]$.

A special case of D-S structure is the classic probability distribution, here each of the B_i is a singleton, $B_i = \{x_i\}$ and $m(B_i) = p_i$, the probability of x_i. This is referred to as a Bayesian belief structure. In this case $Bel(A) = Pl(A) = Prob(A)$. Here $Prob(A) = \sum_{A \cap B_i \neq \emptyset} m(B_i) = \sum_{i,\ x_i \in A} m(B_i)$.

Another special case of belief structure is one in which we have two focal elements $B_1 = B$ and $B_2 = X$. In this case we have $m(B_1) = \alpha$ and $m(B_2) = 1 - \alpha$. This corresponds to a situation where we know that the probability that the value of V lies in B is at least α.

In the case when we have no knowledge about the value of V other than that it lies in the set X we can represent this with a belief structure having one focal element $B = X$ where $m(B) = 1$. We point out the difference between this and the usual probabilistic assumption of "no-knowledge" where we assume for all x_i that $p_i = 1/N$ where N is the number of elements in X. We note that in the case of this usual assumption actually we are not assuming "no knowledge" but we are assuming the knowledge that all elements have the same probability, this is not the case when we use a belief structure described above.

A very important type of information about an uncertain variable is possibilistic uncertainty [14]. We can also represent this using a belief structure. Let V be a variable with domain $X = \{x, \ldots, x_n\}$. Assume we have a possibility distribution Π expressing our information about the uncertain value of V where π_j is the possibility that $V = x_j$. For this type of distribution there exists at least one x_K such that $\pi_K = 1$. Without loss of generality we shall assume that the indexing of the elements in X is such that $\pi_i \geqslant \pi_j$ for $i \leqslant j$. In the case we have that $\pi_1 = 1$. It is well known that we can represent this information using a Dempster–Shafer belief structure m with focal elements $F_j = \{x_1, \ldots, x_j\}$ for $j = 1$ to n with $m(F_j) = \pi_j - \pi_{j+1}$ using the convention that $\pi_{n+1} = 0$. We see here that the F_j are nested, $F_j \subseteq F_{j+1}$. This kind of belief structure was called a consonant belief structure by Shafer [3]. We recall that for any subset A we have $Pl(A) = \sum_{j, A \cap F_j \neq \emptyset} m(F_j)$ and $Bel(A) = \sum_{j, F_j \subseteq A}^{n} m(F_j)$. We see in the case of nested focal elements that

$$Pl(A) = \sum_{j=K}^{n} m(F_j)$$

where F_K is the smallest focal element that intersects with A. Similarly

$$Bel(A) = \sum_{j=1}^{r} m(F_j)$$

where F_r is the largest focal element contained in A.