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Partial order relation for approximation operators in covering based rough sets

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ABSTRACT

Covering based rough sets are a generalization of classical rough sets, in which the traditional partition of the universe induced by an equivalence relation is replaced by a covering. Many definitions have been proposed for the lower and upper approximations within this setting. In this paper, we recall the most important ones and organize them into sixteen dual pairs. Then, to provide more insight into their structure, we investigate order relationships that hold among the approximation operators. In particular, we study a point-wise partial order for lower (resp., upper) approximation operators, that can be used to compare their respective approximation fineness. We establish the Hasse diagram for the partial order, showing the relationship between any pair of lower (resp., upper) operators, and identifying its minimal and maximal elements.

1. Introduction

Rough sets, introduced by Pawlak [9] in 1982, provide approximations of concepts in the presence of incomplete information. Essentially, rough set analysis makes statements about the membership of some element x of a non-empty universe set U to the concept of which $A \subseteq U$ is a set of examples, based on the indiscernibility between x and the elements of A. In particular, x belongs to the lower approximation of A if all elements indiscernible from x belong to A, and to the upper approximation of A if at least one element indiscernible from x is a member of A. In Pawlak's original proposal, indiscernibility is modeled by an equivalence relation on U.

Since then, many generalizations of rough set theory have been proposed. A first one is to replace the equivalence relation by a general binary relation. In this case, the binary relation determines collections of sets that no longer form a partition of U [6–8,12,41]. This generalization has been used in applications with incomplete information systems and tables with continuous attributes [4,5,24,26,42]. A second generalization is to replace the partition obtained by the equivalence relation with a covering; i.e., a collection of non-empty sets with union equal to U [14,21,25,36,38,40]. In this paper, we focus on the latter generalization, called covering based rough sets. Some connections between the two generalizations have also been established, for example in [23,29,37,41].

Unlike in classical rough set theory, there is no unique way to define lower and upper approximation operators in covering based rough set theory. In fact, different equivalent characterizations of the classical approximations cease to be

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equivalent when the partition is generalized by a covering. Based on this observation, in [25], Yao and Yao considered twenty pairs of covering based lower and upper approximation operators, where each pair is governed by a duality constraint. Other operators outside this framework appear for example in [21], where Yang and Li present a summary of seven non-dual pairs of approximation operators that were used by Żakowski [27], Pomykala [11], Tsang [15], Zhu [37], Zhu and Wang [39] and Xu and Zhang [20].

In [13], we investigated properties of these operators and some relationships that hold between them, and in particular we proved a characterization of pairs of approximation operators that are at the same time dual and adjoint (i.e., for which there exists a Galois connection). In this paper, we continue this study of covering based approximation operators, focusing on a particular point-wise partial order relation that can be considered among them. Specifically, given two lower approximation operators $\underline{apr}_1$ and $\underline{apr}_2$, and two upper approximation operators $\overline{apr}_1$ and $\overline{apr}_2$, we say that the pair $(\underline{apr}_1, \overline{apr}_1)$ is *finer than* the pair $(\underline{apr}_2, \overline{apr}_2)$ if $\underline{apr}_2(A) \subseteq \underline{apr}_1(A)$ and $\overline{apr}_1(A) \subseteq \overline{apr}_2(A)$ for every set A in U. Clearly, the "is finer than" relation forms a partial order. Moreover, it is very useful in practice, since it helps practitioners in their choice of suitable approximation operators: indeed, a pair of finer operators will allow to approximate a concept more closely from below and from above.

In this paper, we want to establish this partial order for the most commonly used covering based approximation operators, providing an exhaustive evaluation of their pairwise comparability. The remainder of the paper is structured as follows: in Section 2, we present preliminary concepts about classical rough sets, and review lower and upper approximation operators for covering based rough sets that have been proposed in literature. They belong to two main frameworks: the dual framework of Yao and Yao [25], including element based, granule based and system based definitions; and the non-dual framework of Yang and Li [21]. In Section 3, on one hand, we reduce the number of operators by proving some equivalences between them, and on the other hand, we consider some new ones which emerge as duals of the approximation operators considered by Yang and Li. This gives us sixteen pairs of dual and distinct approximation operators. We also list their most important theoretical properties. Section 4 evaluates the fineness order, first for each group of operators separately, and then for all the operators jointly. The central result of our analysis is a Hasse diagram positioning the 16 lower (resp., upper) approximation operators according to the fineness order. We also show the orders for subsets of operators satisfying particular properties, like adjointness and being a meet/join-morphism. Section 5 presents some conclusions and outlines future work.

2. Preliminaries

Throughout this paper, we will assume that U is a finite and non-empty set; $\mathscr{P}(U)$ represents the collection of subsets of U.

2.1. Rough sets

In Pawlak's proposal [9] of rough sets, an approximation space is an ordered pair (U, R), where R is an equivalence relation on U. Yao and Yao [25] consider three different, equivalent ways to define lower and upper approximation operators which are recalled below.

If (U, R) is an approximation space, for each $A \subseteq U$, the *element based*lower and upper approximations of A by R are defined by:

$$\underline{apr}(A) = \{x \in U : [x]_R \subseteq A\} \tag{1}$$
$$\overline{apr}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\} \tag{2}$$

where $[x]_R$ is the equivalence class of x. On the other hand, the *granule based* lower and upper approximations are defined by:

$$\underline{apr}(A) = \bigcup\{[x]_R \in U/R : [x]_R \subseteq A\} \tag{3}$$
$$\overline{apr}(A) = \bigcup\{[x]_R \in U/R : [x]_R \cap A \neq \emptyset\} \tag{4}$$

Finally, the *system based* approximations are obtained from the σ-algebra $\sigma(U/R)$, generated from the equivalence classes, by adding the empty set and making it closed under set union:

$$\underline{apr}(A) = \bigcup\{X \in \sigma(U/R) : X \subseteq A\} \tag{5}$$
$$\overline{apr}(A) = \bigcap\{X \in \sigma(U/R) : X \supseteq A\} \tag{6}$$

When R is not an equivalence relation (U/R is not a partition), these definitions are no longer equivalent. This has inspired the various proposals for covering based approximation operators in the following subsection.

2.2. Covering based rough sets

Covering based rough sets were proposed to extend the range of applications of rough set theory. The basic idea is to replace the partition corresponding to an approximation space by a covering.