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Sequential composition of linear systems' clans

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ABSTRACT

When using Petri nets to investigate deadlock control, structural analysis techniques are applied, which are based on solving systems of linear algebraic equations. To gain an extra computational speed-up when solving sparse linear systems, we examine a sequential clan-composition process, represented by a weighted graph. The system decomposition into clans is represented by a weighted graph. The comparative analysis of sequential composition for subgraphs and edges (pairwise) is presented. The task of constructing a sequence of systems of lower dimension is called an optimal collapse of a weighted graph; the question whether it is NP-complete remains open. Upper and lower bounds for the collapse width, corresponding to the maximal dimension of systems, are derived. A heuristic greedy algorithm of (quasi) optimal collapse is presented and validated statistically. The technique is applicable for solving sparse systems over arbitrary rings (fields) with sign.

1. Introduction

Solving systems of linear algebraic equations and inequalities plays a key role in the structural analysis techniques of Petri nets for deadlock control [5,16,19,21,22,29,36], and Petri net models of flexible manufacturing systems (FMS) are extensively studied in the literature [6,7,14,15,17,31]. Using these techniques, a system, containing equations and inequalities, is transformed into a purely equational system (possibly with additional variables), but the high dimensionality of models of real-life manufacturing systems makes their analysis difficult from the computational point of view because it implies solving complex Diophantine systems [16,21,32].

Indeed, many tasks of modern science and technology can be reduced to solving systems of linear equations and inequalities [28]. To solve systems *in fields*, a variety of direct and iterative methods have been developed based on system matrix factorization into diagonal, upper- and lower-triangular matrices [9,24,35]. The time complexity of solutions is polynomial, chiefly cubic, with respect to the number of the matrix rows (columns). To deal with sparse matrices, special techniques of decomposition were introduced based on permutations of the matrix rows and columns [9,27] and the problem is often reduced to one of graph decomposition [11,13,37].

Solving systems *in rings*, for instance Diophantine systems, is based on Smith and Hermite normal forms and unimodular transformations of the matrix [8,10,12,25]. The best known methods use time that is a fourth-degree polynomial [10,25] but as mentioned in [12] the estimations should be exponential when GCD operation complexity is involved.

Some domains require solving systems *over rings in monoids*, for instance a Diophantine system in nonnegative integers. Such tasks arise in Petri net theory [23] and lead to developing special methods [3,26] which can be traced to early Fourier works and collate coefficients with opposite signs. The number of basis solutions can grow rapidly and the space complexity is estimated as exponential [26]. This fact hampers the application of matrix methods for real-life systems analysis.

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Table 1
The legend of matrix A decomposition into clans.

Clans/Variables	X^0	$\overset{\frown}{X}^1$	$\overset{\frown}{X}^2$	...	$\overset{\frown}{X}^k$
C^1	$A^{0,1}$	$\overset{\frown}{A}^1$	0	...	0
C^2	$A^{0,2}$	0	$\overset{\frown}{A}^2$	...	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
C^k	$A^{0,k}$	0	0	...	$\overset{\frown}{A}^k$

In [33], a decomposition of a system was presented that is entirely based on the sign concept. It factors the source system into a set of subsystems called *clans*; the obtained block structure of the matrix is a union of a block column and diagonal block matrix. Solving systems for each clan and then a composition system gives a considerable speed-up. The composition of clans can be applied for solving systems in arbitrary rings (fields) with sign as well.

In [33] we implemented the simultaneous composition of all clans in the source system, which meant solving the composition system as a monolithic entity. In some cases, the size of this composition system is considerable, which results in high total complexity when solving the source system. The goal of the present work is to formalize the task of sequentially composing a linear system's clans and to design efficient algorithms for its solution. The results can be considered a generalization to arbitrary matrices with sign of the technique of Petri net decomposition into functional subnets and their subsequent re-composition [34].

2. Solving linear systems via composition of their clans

We study a system of linear algebraic equations of the following form:

$$A \cdot \bar{x} = \bar{b}, \tag{1}$$

where A is a matrix of coefficients of dimension $m \times n$, $\bar{x}$ is a vector-column of unknowns of dimension n and $\bar{b}$ is a vector-column of free terms of dimension m. When $\bar{b} = 0$, the system is said to be homogeneous, and when $\bar{b} \neq 0$, heterogeneous. As in [33], the sets of coefficients, free terms, and unknowns are not specified exactly. We only suppose that the algebraic structure of matrix A elements includes a sign concept and that there is a known method for solving system (1) and representing its general solution in the following form:

$$\bar{x} = \bar{x}' + G \cdot \bar{y}, \tag{2}$$

where $G \cdot \bar{y}$ is a general solution of the corresponding homogenous system, and $\bar{x}'$ is a (minimal) particular solution of the heterogeneous system (1).

The compositional method for solving system (1), presented in [33], consists of the following stages:

I. Decomposition of a system into clans.
II. Finding a general solution for each clan.
III. Composition of clans.

Recall that, a *clan* is a subset of equations formed as a transitive closure of a *near relation*; two equations are *near* if they contain a variable with the same sign; variables, which enter only one clan, are called *internal variables* of a clan; variables which enter more than one clan are called *contact variables* for those clans. We have previously shown that contact variables always belong to exactly two clans and enter them with opposite signs. As a result of decomposition into clans, the following block form of matrix A is obtained:

$$A = \begin{Vmatrix} A^{0,1} & \overset{\frown}{A}^1 & 0 & 0 & 0 \\ A^{0,2} & 0 & \overset{\frown}{A}^2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{0,k} & 0 & 0 & 0 & \overset{\frown}{A}^k \end{Vmatrix}.$$

With respect to clans and subsets of variables, the blocks of matrix A are described in Table 1, where C^i denotes clans, $\overset{\frown}{X}^i$ – internal variables of clans, X^0 – contact variables. Nonzero columns of submatrix $A^{0,i}$ form a set $\overset{\smile}{X}^i \subset X^0$ of contact variables of the clan C^i and a pair $(\overset{\smile}{X}^i, \overset{\frown}{X}^i)$ defines all variables of a clan C^i with nonzero coefficients. We have previously shown that an arbitrary contact variable $x_j \in X^0$ belongs to precisely two clans and enters these clans with opposite signs. In what follows, the notations $I(x)$ and $O(x)$ denote input and output clans, respectively, of a contact variable x. Similarly,