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Rough sets based on complete completely distributive lattice

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ABSTRACT

In this paper, a pair of rough approximation operators on a complete completely distributive (CCD) lattice based on an ordinary binary relation is defined. This kind of rough sets can be seen as a unified framework for the study of rough sets based on ordinary binary relations, rough fuzzy sets and interval-valued rough fuzzy set. Moreover, depending on classes of binary relations, this paper defines several classes of rough sets on CCD lattices and investigates properties of these classes. Finally, two generalized rough set models on two CCD lattices are given at the end of this paper.

1. Introduction

Rough sets theory, originally proposed by Pawlak [19], is an important tool for dealing with incomplete and insufficient information. It has found real applications in such areas of science as information analysis, data mining, linguistics, medicine and so on. A key concept in Pawlak's rough sets is the equivalence relation (indiscernibility relation) or partition. But in some situations, equivalence relations are too restrictive for many applications. To address this issue, a lot of attempts have been made to extend the equivalence relation to more general relations. One generalization approach is to relax equivalence relations to tolerance relations [18,23] or similarity relations [21,24], or even arbitrary binary relations [14,26]. Another generalization approach is built on the partition of disjoint granules (equivalence classes) with a cover consisting of intersecting granules [1,3,17]. Since fuzzy sets theory and rough sets theory are two important tools to solve imprecise and incomplete, it is a natural idea to combine them. The concepts of rough fuzzy sets and fuzzy rough sets were first proposed in 1990 [5,6]. Then many researchers have studied rough sets in this field, see [10,11,14,15]. Furthermore, the universe of rough sets were extended from one to two, such as generalized rough sets [27], generalized rough fuzzy set [13], fuzzy rough sets [16,25] and generalized interval-valued fuzzy rough sets [30]. Recently, it becomes an interesting research area to construct framework for the study of rough sets. Csajbók et al. [4] proposed approximation framework to describe the characterization of rough set models. Chen et al. [3] defined a pair rough approximations. In this way, rough sets and fuzzy rough sets were brought into a unified framework. Šostak [22] introduced an M-approximate systems, which developed a framework to generalize rough set theory.

The crisp power set of a universe is an atomic Boolean lattice, and in the Pawlak's rough sets equivalence classes are the elements of this power set. From the aspect of the order relation on a lattice, the lower approximation of a subset *A* of the universe is some element of its power set lattice which is not larger than *A*, and the upper approximation of the subset *A* is some element not smaller than *A*. Then a natural ideal in a lattice appears. Can we study rough sets from the aspect of the

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lattice theory? That means if we take a lattice as the universe, what kinds of elements can be defined as the lower or upper approximations. Using a mapping from the power set $\mathscr{P}(A)$ to A, where A is a given set, Järvinen [12] proposed a more general framework for the study of approximation based on atomic Boolean lattice $\mathscr{P}(A)$. Employing the concept of partitions or covers, Mordeson [17] defined approximation operators on the power set of a given set; Qi and Liu [20] proposed a rough set and a generalized rough set on Boolean lattice. In addition, the fuzzy power set of a universe is a molecular lattice, as known as a CCD lattice, in which every non-zero element is a supermum of some moleculars. Therefore, in order to further research the different kinds of generalizations of rough set models and bring them into a unified framework, lattices become a wider mathematical foundation applied to construct rough set models. Chen et al. [3] defined rough approximations based on covers of a CCD lattice. The topological properties and structure of these rough sets on CCD lattice were presented in [7,8].

From the above we conclude that CCD lattices are the most general structure to construct rough sets until now, in particular the authors in [3] have defined. We know that the notion of a relation, which is an important concept in mathematics, is more widely used in many fields, such as data analysis and knowledge discover, than the concept of a cover. It is a key notion in Pawlak's rough sets. The extended rough sets based on binary relation also play important parts in rough sets theory. So in this paper, we select an arbitrary binary relation on a CCD lattice to define rough sets on the CCD lattice, which can be seen as a unified framework for the research of rough sets based on ordinary binary relations, rough fuzzy sets and interval-valued rough fuzzy sets. We discussed the properties of rough sets and propose generalized rough sets on the CCD lattice. The rough sets defined on a CCD lattice are the generalization of rough sets defined on power lattices [12] and Boolean lattices [20]. The rough sets defined in [3] are special cases of the rough sets as they are defined in this paper, when the cover is reduced. However, generally these two kinds of rough sets cannot replace one another.

The paper is organized as follows. In Section 2, we briefly recall some important concepts in the lattice theory. Section 3 presents definitions and basic properties of the rough sets defined by means of arbitrary binary relation. In Section 4, we discuss several classes of rough sets and their properties. The generalized rough sets based on two universes are defined in Section 5. The paper is completed by conclusions in Section 6.

2. Preliminaries

In our work, the complete completely distributive lattice, called CCD lattice for short, plays the fundamental role. In this section, we recall some basic concepts of CCD lattice to be used in this paper.

A CCD lattice $(L, \leqslant, \vee, \wedge)$ is a complete lattice and satisfies the completely distributivity laws

$$\bigvee_{i\in I}\left(\bigwedge_{j\in J_i} a_{ij}\right) = \bigwedge_{f\in\prod_{i\in I}J_i}\left(\bigvee_{i\in I} a_{if(i)}\right),$$

$$\bigwedge_{i\in I}\left(\bigvee_{j\in J_i} a_{ij}\right) = \bigvee_{f\in\prod_{i\in I}J_i}\left(\bigwedge_{i\in I} a_{if(i)}\right),$$

where I and J_i are nonempty index sets and $a_{ij} \in L$. A non-zero element e is said to be join-irreducible if for $a, b \in L$, $e \leqslant a \vee b$ implies $e \leqslant a$ or $e \leqslant b$. The set of all join-irreducible elements of L is denoted as $M(L)$. It is well known that each non-zero element of a CCD lattice is a supermum of some join-irreducible elements [9], i.e. $\forall a \in L - \{0\}$, there exists a subset A of $M(L)$, such that $a = \vee\{e : e \in A\}$.

Example 1. Here are some popular examples of CCD lattices.

(1) Let $L = [0, 1]$. L is a CCD lattice with natural order and $M(L) = L - \{0\}$.
(2) (power set lattice) Let X be a nonempty set and $\mathscr{P}(X)$ be the power set of X. $(\mathscr{P}(X), \cap, \cup, \emptyset, X)$ is a CCD lattice called a power set lattice and $M(L) = \{\{x\} : x \in X\}$.
(3) (fuzzy power set lattice) Let X be a nonempty set and $\mathscr{F}(X)$ be the collection of all fuzzy sets of X. $(\mathscr{F}(X), \cap, \cup, \emptyset, X)$ is a CCD lattice called a fuzzy power set lattice and $M(\mathscr{F}(X)) = \{\{x_\lambda\} : x \in X, \lambda \in (0, 1]\}$, where

$$x_\lambda(y) = \begin{cases} \lambda, & y = x \\ 0, & y \neq x. \end{cases}$$

(4) (interval-valued fuzzy set lattice) Let X be a nonempty set. The mapping $A : X \to I_{[0,1]}$ is called an interval-valued fuzzy set on X [28,29], where

$$I_{[0,1]} = \{[a^-, a^+] : 0 \leqslant a^- \leqslant a^+ \leqslant 1, a^-, a^+ \in \mathbf{R}\}.$$

The collection of all interval-valued fuzzy sets on X is denoted as $\mathscr{F}_I(X)$.

In fact, for every $A \in \mathscr{F}_I(X)$ and $x \in X$, we have $A(x) = [A^-(x), A^+(x)]$, where $A^-, A^+ \in \mathscr{F}(X)$. Similarly to the fuzzy sets, the operators $\subseteq, \cap, \cup$ on interval-valued fuzzy sets are defined as follows, for every $A, B \in \mathscr{F}_I(X)$,