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Some results on a transformation of copulas and quasi-copulas

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ABSTRACT

This paper provides some results including invariance properties, dependence measures, convexity properties and tail dependence on a transformation of copulas and quasi-copulas.

1. Introduction

A (bivariate) copula is a function C: $[0,1]^2 \to [0,1]$ such that (C1) $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in [0,1]$, and (C2) $V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0$ for all u_1, u_2, v_1, v_2 in $[0,1]$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. Copulas have proved to be a useful tool in the construction of multivariate distribution functions. In fact, in view of Sklar's Theorem [22], the joint distribution H of a pair of random variables—defined on a common probability space $(\Omega, \mathbb{P})$—and the corresponding marginal distributions F and G are linked by a copula C in the following manner: $H(x,y) = C(F(x),G(y))$ for all x,y in $[-\infty,\infty]$. For a complete review of this concept and some of its applications see [3,12,18]. Let Π denote the copula for independent random variables, i.e., $\Pi(u,v) = uv$ for all (u,v) in $[0,1]^2$. Every copula C satisfies the following inequalities:

$$\max(u+v-1,0) = W(u,v) \leqslant C(u,v) \leqslant M(u,v) = \min(u,v), \quad \forall (u,v) \in [0,1]^2,$$

where M and W are themselves copulas. Various procedures to construct copulas have been proposed in the literature (e.g., see [13,18]). Recently, some authors provided construction methods from the class of copulas to itself, or from a more general class of functions to another (e.g., see [2,4,6,14,17]).

For a given copula D and α, $\beta \in [0,1]$, consider the function $C_{\alpha,\beta}[D]$ defined by

$$C_{\alpha,\beta}[D](u,v) = u^{1-\alpha} v^{1-\beta} D(u^{\alpha}, v^{\beta}), \tag{1}$$

for every $(u,v) \in [0,1]^2$. This structure is, in fact, a copula and first appeared in [10,15] as a mechanism for generating asymmetric copulas. Indeed, if the pair (U_1,U_2) and (V_1,V_2) are two independent vectors of uniform (0,1) random variables with associated copulas D and Π, respectively, then $C_{\alpha,\beta}[D]$ is the joint distribution of the pair (U,V) defined by

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$$U = \max\{U_1^{1/\alpha}, V_1^{1/(1-\alpha)}\}, \quad V = \max\{U_2^{1/\beta}, V_2^{1/(1-\beta)}\}.$$

A multivariate generalization of (1) is given in [16]. Observe that $C_{0,0}[D] = \Pi$, $C_{1,1}[D] = D$ and

$$C_{\alpha,\beta}[M](u,v) = \min(u^{1-\alpha}v, uv^{1-\beta}), \tag{2}$$

is the Marshal–Olkin (which we denote by MO) family of copulas [18]. To the best of our knowledge properties of the family (1) had not been thoroughly investigated. The aim of this paper is to provide some results for this family, including invariance properties, dependence measures, convexity properties and tail dependence (Section 2). In Section 3 we study the case in which D is a proper quasi-copula.

2. Properties

In this section we provide some results including invariance properties, dependence measures, convexity properties and tail dependence on the family (1).

2.1. Invariance

We say that a given copula D is $C_{\alpha,\beta}$ *invariant* under (1) if $C_{\alpha,\beta}[D] = D$ for every $\alpha, \beta \in [0,1]$. First note that the transformation (1) is "unique", in the sense that if D_1 and D_2 are two copulas such that $C_{\alpha,\beta}[D_1] = C_{\alpha,\beta}[D_2]$ for every $\alpha, \beta \in [0,1]$, then $D_1 = D_2$. Although we do not find a pattern to describe the class of $C_{\alpha,\beta}$ invariant copulas, we note that, for example, the Gumbel–Barnett family of copulas [18], defined by

$$D_\theta(u,v) = uve^{-\theta \ln(u)\ln(v)}, \quad 0 \leqslant \theta \leqslant 1,$$

satisfies $C_{\alpha,\beta}[D_\theta] = D_{\alpha\beta\theta}$. The following example shows the invariance property of (1) in the class of extreme value copulas.

Example 1. Consider the family of extreme value copulas [20]

$$D_A(u,v) = \exp\left[\ln(uv)A\left(\frac{\ln(u)}{\ln(uv)}\right)\right],$$

with the dependence function A: $[0,1] \to [0,1/2]$, satisfying $A(0) = A(1) = 1$ and $\max(t, 1-t) \leqslant A(t) \leqslant 1$. Under the construction (1) we see that $C_{\alpha,\beta}[D_A] = D_{A_{\alpha,\beta}}$, where

$$A_{\alpha,\beta}(t) = (1-\alpha)t + (1-\beta)(1-t) + (\alpha t + \beta(1-t))A\left(\frac{\alpha t}{\alpha t + \beta(1-t)}\right).$$

We note that the MO family of copulas (2) with the parameters λ and γ, is a member of the extreme value class of copulas with the dependence function $A(t) = 1 - \min(\lambda t, \gamma(1-t))$ and the resulting copula under (1) is a gain a MO copula with the new parameters $\alpha\lambda$ and $\beta\gamma$.

We also have the following result, whose proof is immediate and we omit it.

Proposition 1. *For each copula D and α_1, β_1, α_2, $\beta_2 \in [0,1]$, the copula given by (1) satisfies the stability property $C_{\alpha_2,\beta_2}[C_{\alpha_1,\beta_1}[D]] = C_{\alpha_1\alpha_2,\beta_1\beta_2}[D]$.*

Example 2. Consider the FGM family of copulas, defined by $D_\theta(u,v) = uv[1 + \theta(1-u)(1-v)]$ for all $(u,v) \in [0,1]^2$, with $\theta \in [-1,1]$. Then we have that

$$C_{\alpha,\beta}[D_\theta](u,v) = uv[1 + \theta(1-u^\alpha)(1-v^\beta)], \quad (u,v) \in [0,1]^2, \tag{3}$$

which is an extension of the FGM family of copulas studied in [11]. The resulting family (3) is stable under (1).

For any convex linear combination of copulas (which is again a copula [18]) the following result is immediate.

Proposition 2. *Let $D_\lambda(u,v) = \lambda D_1(u,v) + (1-\lambda)D_2(u,v)$, $\lambda \in [0,1]$, be a linear convex combination of two copulas D_1 and D_2. Then*

$$C_{\alpha,\beta}[D_\lambda](u,v) = \lambda C_{\alpha,\beta}[D_1](u,v) + (1-\lambda)C_{\alpha,\beta}[D_2](u,v).$$

The following result shows the invariance of the copulas M, Π and W.

Proposition 3. *Given a copula D, and the transformation defined by (1) for $\alpha, \beta \in (0,1]$, we have:*

(i) $C_{\alpha,\beta}[D] = M$ if, and only if, $D = M$ and $\alpha = \beta = 1$.
(ii) $C_{\alpha,\beta}[D] = W$ if, and only if, $D = W$ and $\alpha = \beta = 1$.
(iii) $C_{\alpha,\beta}[D] = \Pi$ if, and only if, $D = \Pi$.