Information Sciences 257 (2014) 183–192

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Atoms of weakly null-additive monotone measures and integrals

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ARTICLE INFO

Article history:
Received 18 January 2013
Received in revised form 29 July 2013
Accepted 3 September 2013
Available online 11 September 2013

Keywords:
Monotone measure
Atom
Weak null-additivity
Regularity
Sugeno integral
Choquet integral

ABSTRACT

In this paper, we prove some properties of atoms of weakly null-additive monotone measures. By using the regularity and weak null-additivity, a sin-gleton characterization of atoms of monotone measures on a metric space is shown. It is a generalization of previous results obtained by Pap. The calculation of the Sugeno integral and the Choquet integral over an atom is also presented, respectively. Similar results for recently introduced universal integral are also given. Following these results, it is shown that the Sugeno integral and the Choquet integral over an atom of monotone measure is maxitive linear and standard linear, respectively. Convergence theorems for the Sugeno integral and the Choquet integral over an atom of a monotone measure are also shown.

1. Introduction

An atom of a measure is an important concept in the classical measure theory [6] and probability theory. This concept was generalized in non-additive measure theory. The atoms for submeasures on locally compact Hausdorff spaces were discussed by Dobrakov [4]. In 1991, Suzuki [24] first introduced the concept of an atom of fuzzy measures (non-negative monotone set functions with continuity from below and above and vanishing at $\emptyset$), and investigated some analytical properties of atoms of fuzzy measures. Further research on this matter was made by Pap [19–22], Jiang and Suzuki [7,8], Li et al. [15], Wu and Sun [26], and Kawabe [9,11]. In [21] Pap showed a singleton characterization of atoms of regular null-additive monotone set functions, i.e., if a non-negative monotone set function μ is regular and null-additive, then every atom of μ has an outstanding property that all the mass of the atom is concentrated on a single point in the atom.

In this paper, we shall further investigate some properties of atoms of weakly null-additive monotone measures on metric spaces. We shall show that if a regular monotone measure is weakly null-additive, then the previous results obtained by Pap [21] remain valid. This fact makes easy the calculation of the Sugeno integral and the Choquet integral over an atom of a monotone measure which is regular and countably weakly null-additive. Following these results, it is shown that the Sugeno integral and the Choquet integral over an atom of a monotone measure is maxitive linear and standard linear (cf.[17]),

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0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.09.013

respectively. The convergence theorems for the Sugeno integral and the Choquet integral over an atom of a monotone measure are shown, too.

2. Preliminaries

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X, and $\mathbb{N}$ denote the set of all positive integers. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

A set function $\mu : \mathcal{F} \to [0, \infty]$ is said to be *continuous from below* [3], if $\lim_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *continuous from above* [3], if $\lim_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $A_n \searrow A$ and there exists n_0 with $\mu(A_{n_0}) < \infty$; *continuous*, if μ is continuous from below and above; *order continuous* [21], if $\lim_{n\to\infty}\mu(A_n) = 0$ whenever $A_n \searrow \emptyset$; *exhaustive* [4], if $\lim_{n\to\infty}\mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_{n\in N}$; *strongly order continuous* [13], if $\lim_{n\to\infty}\mu(A_n) = 0$ whenever $A_n \searrow A$ and $\mu(A) = 0$.

A set function μ is called *finite*, if $\mu(X) < \infty$; *σ-finite* [21], if there exists a sequence $\{X_n\} \subset \mathcal{F}$ such that

$$X_1 \subset X_2 \subset \cdots, \quad X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad \mu(X_n) < \infty \ (n = 1, 2, \ldots).$$

Definition 2.1 [27]. A *monotone measure* on $\mathcal{F}$ is an extended real valued set function $\mu : \mathcal{F} \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$; (vanishing at $\emptyset$).
(2) $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity).

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space [21,27].

In this paper, we always assume that μ is a monotone measure on $\mathcal{F}$.

Definition 2.2 [27]. μ is called weakly null-additive, if for any $E,\ F \in \mathcal{F}$,

$$\mu(E) = \mu(F) = 0 \Rightarrow \mu(E \cup F) = 0.$$

In the following we recall several concepts related to the weak null-additivity of non-negative set functions.

Definition 2.3.

(i) μ is said to be *null-additive*, if $\mu(E \cup F) = \mu(E)$ whenever $E, F \in \mathcal{F}$ and $\mu(F) = 0$, see [27].
(ii) μ is said to be *weakly asymptotic null-additive*, if $\mu(E_n \cup F_n) \searrow 0$ whenever $\{E_n\}$ and $\{F_n\}$ are decreasing sequences with $\mu(E_n) \searrow 0$ and $\mu(F_n) \searrow 0$, see [10].
(iii) μ is said to have *pseudometric generating property* (for short, (p.g.p.)), if $\mu(E_n \cup F_n) \to 0$ whenever the sequences $\{E_n\} \subset \mathcal{F}$ and $\{F_n\} \subset \mathcal{F}$ with $\mu(E_n) \to 0$ and $\mu(F_n) \to 0$, see [5].

Obviously, the null-additivity of μ implies weak null-additivity. The pseudometric generating property implies weak asymptotic null-additivity, and the latter implies weak null-additivity.

Definition 2.4 [14]. μ is called *countably weakly null-additive*, if for any $\{A_n\}_{n\in N} \subset \mathcal{F}$,

$$\mu(A_n) = 0, \quad \forall n \geqslant 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0.$$

Definition 2.5 [1]. μ is called *null-continuous*, if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}_{n\in N} \subset \mathcal{F}$ such that $\mu(A_n) = 0,\ n = 1, 2, \ldots$.

We give some relationships among the above introduced properties.

Proposition 2.6. *μ is countably weakly null-additive if and only if μ is both weakly null-additive and null-continuous.*

Proposition 2.7 [1]. *If μ is weakly null-additive and strongly order continuous, then it is null-continuous.*