Information Sciences 257 (2014) 193–209

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Which weighted circulant networks have perfect state transfer? ☆

Milan Bašić

Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia

ARTICLE INFO

Article history:
Received 9 February 2011
Received in revised form 21 March 2013
Accepted 1 September 2013
Available online 11 September 2013

Keywords:
Circulant network
Quantum system
Perfect state transfer
Weighted graph

ABSTRACT

The question of perfect state transfer existence in quantum spin networks based on weighted graphs has been recently presented by many authors. We give a simple condition for characterizing weighted circulant graphs allowing perfect state transfer in terms of their eigenvalues. This is done by extending the results about quantum periodicity existence in the networks obtained by Saxena, Severini and Shparlinski and characterizing integral graphs among weighted circulant graphs. Finally, classes of weighted circulant graphs supporting perfect state transfer are found. These classes completely cover the class of circulant graphs having perfect state transfer in the unweighted case. In fact, we show that there exists an weighted integral circulant graph with n vertices having perfect state transfer if and only if n is even. Moreover we prove the nonexistence of perfect state transfer for several other classes of weighted integral circulant graphs of even order.

1. Introduction

The transfer of a quantum state from one location to another is a crucial ingredient for many quantum information processing protocols. There are various physical systems that can serve as quantum channels, one of them being a quantum spin network. These networks consist of n qubits where some pairs of qubits are coupled via XY-interaction. The perfect transfer of quantum states from one qubit to another in such networks, was first considered in [12,14]. There are two special qubits A and B representing the input and output qubit, respectively. The transfer is implemented by setting the qubit A in a prescribed quantum state and by retrieving the state from the output qubit B after some time. A transfer is called *perfect state transfer (PST)* (transfer with unit fidelity) if the initial state of the qubit A and then final state of the qubit B are equal due to the local phase rotation. If the previous condition holds for $A = B$, the network is *periodic* at A. A network is *periodic* if it is periodic at each qubit A. For such networks, periodicity is a necessary condition for the perfect state transfer existence.

Every quantum spin network with fixed nearest-neighbor couplings is uniquely described by an undirected graph G on a vertex set $V(G) = \{1, 2, \ldots, n\}$. The edges of the graph G specify which qubits are coupled. In other words, there is an edge between vertices i and j if ith and jth qubit are coupled.

In [12] a simple XY coupling is considered such that the Hamiltonian of the system has the form

$$H_G = \frac{1}{2} \sum_{(i,j)\in E(G)} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y.$$

☆ The author gratefully acknowledges support from the research project 174013 of the Serbian Ministry of Science.
E-mail address: basic_milan@yahoo.com

0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.09.002

and σ_i^x, σ_i^y and σ_i^z are Pauli matrices acting on ith qubit. The standard basis chosen for an individual qubit is $\{|0\rangle, |1\rangle\}$ and it is assumed that all spins initially point down ($|0\rangle$) along the prescribed z axis. In other words, the initial state of the network is $|\underline{0}\rangle = |0_A0\dots00_B\rangle$. This is an eigenstate of Hamiltonian H_G corresponding to zero energy. The Hilbert space $\mathcal{H}_G$ associated to a network is spanned by the vectors $|e_1,e_2,\dots,e_n\rangle$ where $e_i \in \{0,1\}$ and, therefore, its dimension is 2^n.

The process of transmitting a quantum state from A to B begins with the creation of the initial state $\alpha|0_A0\dots00_B\rangle + \beta|1_A0\dots00_B\rangle$ of the network. Since $|\underline{0}\rangle$ is a zero-energy eigenstate of H_G, the coefficient α will not change in time. Since the operator of total z component of the spin $\sigma^z_{tot} = \sum_{i=1}^n \sigma_i^z$ commutes with H_G, state $|1_A0\dots00_B\rangle$ must evolve into a superposition of the states $|i\rangle = |0\dots01_i0,\dots,0\rangle$ for $i = 1,\dots,n$. Denote by $\mathcal{S}_G$ the subspace of $\mathcal{H}_G$ spanned by the vectors $|i\rangle$, $i = 1,\dots,n$. Hence, the initial state of network evolves in time t into the state

$$\alpha|\underline{0}\rangle + \sum_{i=1}^{n} \beta_i(t)|i\rangle \in \mathcal{S}_G.$$

The previous equation shows that system dynamics is completely determined by the evolution in n-dimensional space $\mathcal{S}_G$. The restriction of the Hamiltonian H_G to the subspace $\mathcal{S}_G$ is an $n \times n$ matrix identical to the adjacency matrix A_G of the graph G.

Thus, the time evolution operator can be written in the form $F(t) = \exp(\imath A_G t)$. The matrix exponential $\exp(M)$ is defined as usual

$$\exp(M) = \sum_{n=0}^{+\infty} \frac{1}{n!} M^n.$$

Perfect state transfer (PST) between different vertices (qubits) a and b ($1 \leqslant a,b \leqslant n$) is obtained in time τ, if $\langle a|F(t)|b\rangle = |F(\tau)_{ab}| = 1$. Now formally, the graph (network) is periodic at a if $|F(\tau)_{aa}| = 1$ for some τ.

There are many problems whose characteristics are appropriate for new additional tools to be discovered using some quantum algorithmic ideas. Namely, many complex and nonlinear systems with complicated dynamics require mathematical models which are capable to control them. Most often, the models use complex logic which substantially increase the difficulty in their designing. The problems which arise from such setting are mostly NP-complete and require optimal solutions for the purpose of physically implementation of the complex systems such as multi-agent systems [24]. Many recent papers proposed an evolutionary methodology based on the principles of quantum computing to optimize the parameters of control models. For example, the quantum evolutionary algorithms were applied to find the best parameter matrix of linear matrix inequality of fuzzy logic model [31]. The same methodology was used in solving the NP-complete problem of state encoding in the design process of finite state machines [25,26]. Moreover, instead of traditionally adiabatic evolution algorithm that cannot guarantee to resolve NP-complete problem for computing initial parameters for support vector regression, Chang et al. proposed quantum-neuron-based approach that can break NP-complete problem [11].

Generally, as in the above cases, a natural way to discover new quantum algorithmic ideas is to adapt a classical one to the quantum model. The quantum analogue of classical random walks (for short quantum walks) has been studied in recent papers showing that they have better performance compered to the classical walks on some topologies such as hypercubes. As we may conveniently view PST problem in the context of quantum walks (the quantum walk amplitude at some time must be of unit magnitude), our goal is to show further qualitative features of the quantum walks on circulant topologies.

In a recent work of Saxena, Severini and Shparlinski [28], circulant graphs were proposed as potential candidates for modeling quantum spin networks enabling the perfect state transfer between antipodal sites in a network. It was shown that a quantum network whose hamiltonian is identical to the adjacency matrix of a circulant graph is periodic if and only if all eigenvalues of the graph are integers (that is, the graph is integral). Therefore, circulant graphs having PST must be *integral circulant graphs*. Circulant graphs are also an important class of interconnection networks in parallel and distributed computing (see [19]).

Integral circulant graphs were first characterized by So [29]. Some properties of integral circulant graphs, including the bound of the number of vertices, diameter and bipartiteness were later studied by [5,21,28,30]. Moreover, integral circulant graphs are a generalization of the well-known class of unitary Cayley graphs. Various properties of unitary Cayley graphs were investigated in some recent papers as the diameter, clique number, chromatic number, eigenvalues and size of the longest induced cycles [7,23]. Integral circulant graphs have found important applications in molecular chemistry for modeling energy–like quantities [20].

The existence of PST for some network topologies has already been considered in the literature. For example, Christandl et al. [12,14] proved that PST occurs in paths of length one and two between their end-vertices and also in Cartesian powers of these graphs between vertices at maximal distance. In the recent paper [17], Godsil constructed a class of distance-regular graphs of diameter three, with PST.

Some research of the existence of PST over circulant topologies was already performed. In [3] authors gave a simple and general characterization of PST existence in integral circulant graphs and in a recent paper [4], complete characterization of integral circulant graph having PST was given. Some other properties of quantum dynamics such as uniform mixing on circulant graphs were studied in [1].

In all known classes of graphs having PST *perfect quantum communication distances* (i.e., the distances between vertices where PST occurs) are considerably small compared to the order of the graph. One idea how to increase these, is to consider networks with fixed but different couplings between qubits. These networks correspond to graphs with weighted adjacency matrices with a Hamiltonian