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Rough sets and matroids from a lattice-theoretic viewpoint

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ABSTRACT

This paper studies rough sets via matroidal approaches from a lattice-theoretic viewpoint. We firstly give a new interpretation of definable sets of Pawlak rough set model, i.e., the set of definable sets defines uniquely a matroid, in which it is the family of open and closed sets. Then we induce two equivalence relations on a given universe based on a matroid defined on this universe. One of the equivalence relations actually is defined on the set of all atoms of a geometric lattice corresponding to the matroid, another is based on the transitivity of circuits. Properties of these two equivalence relations are then studied. Besides, we also investigate the connections between relation-based rough sets and matroids. Finally, we point out that a geometric lattice can induce a series of coverings of a universe, on which the corresponding matroid is defined, and further relations of approximations based on the induced coverings are studied.

1. Introduction

The basic ideas of rough set theory, proposed by Pawlak in 1982 [21], deal with situations in which the objects of a given universe can be identified only within the limits of imprecise data by an indiscernibility relation. The indiscernibility relation enables us to characterize a set of objects by a pair of sets, called the lower and upper approximation of the set of objects. Rough set theory has been successfully applied to many fields, for example, in artificial intelligence, computer science, decision theory, expert systems, operations research, pattern recognition, etc. For a detailed introduction of rough sets, see Pawlak [22] or his survey papers [23–25].

The rough set concept overlaps in many aspects of other mathematical ideas to deal with imprecision and vagueness, in particular with fuzzy sets [9,41], theory of evidence [29] and soft sets [7,8]. Furthermore, structural properties of rough sets in the viewpoints of various mathematical branches have been studied and different interpretations of concepts and notions of rough set theory have been given, such as rough set with topological spaces [14,26,45], algebras [4,19] and orders (lattices) [5,12], which are three fundamental abstract structures suggested by Bourbaki.

The concept of matroid was coined by Whitney in 1935 to study an abstract theory of independence. Matroids appear in various combinatorial and algebraic contexts and have proved to be essential important in many fields [35], particularly in discrete optimization [13]. In recent years, matroids are also generalized and used as a tool to other mathematical branches, such as to lattices [28], fuzzy sets [10,15] and concept lattices [18].

As pointed in [17], matroids appear in various mathematical branches, thus we can give explanations of matroidal structures in different mathematical backgrounds; matroids abstract graphic structure, then we can characterize matroidal

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structures in graph clearly; matroids have a lattice-theoretic characterization, i.e. geometry lattices, which make us can investigate matroidal structures using lattice theory. All these advantages of matroids make it tempting to study rough sets with matroidal approaches. Tsumoto and Tanaka [30,31] first studied the connections of rough sets and matroids, they use matroid theory to understand the differences and similarities among three methods of inductive learning, i.e., AQ, Pawlak's Consistent Rules and ID3. After that for a long time, there is no paper concerning combination of rough sets and matroid theory. Until 2007, Deng [6] used the rank function of matroids to study rough sets, Li and Liu [16] characterized the Pawlak rough set model via matroidal approaches. Many related papers are then published, for example, Zhu and Wang [46] proposed rough matroids based on relations, Wang et al. [32] described attribute reduction using matroid theory. Li and Liu [17] generalized the upper and lower approximation operators to the closure and interior operators of matroids and studied structural properties of Pawlak rough set model via matroidal approaches. Unfortunately, all these papers never refer to geometric lattices, a lattice-language version of matroids. In this paper, we make a try in this aspect and study rough sets by matroidal approaches from a lattice-theoretic viewpoint.

The remainder of this paper is organized as follows. Section 2 presents some fundamental definitions and properties of rough sets, matroids and lattices, which will be need in this paper. In Section 3, we studies properties of definable sets to give new interpretations of Pawlak rough set model. For a universe and a matroid defined on it, we in Section 4 induce two equivalence relations defined on the universe via the matroid. One equivalence relation is based on the famous transitivity theorem of circuits and the other in fact is defined on the set of all atoms of a geometric lattice. Approximations based on the two equivalence relations are also compared. In Section 5, we prove that elements of every level of a geometric lattice can form a covering of a universe, on which the corresponding matroid is defined and investigate the properties of approximations based on the induced coverings. A concluding remark is given in the last section.

2. Preliminaries

In this section, we introduce some basic concepts and results on Pawlak rough sets, matroids and lattices, most of which are from [22], [20] and [1] respectively. We shall assume that, unless otherwise specified, all sets (posets) in this paper are finite.

2.1. Relations and Pawlak rough sets

Let A and B be two finite sets, a binary relation R from A to B is a subset of the Cartesian product $A \times B = \{(a, b) \mid a \in A$ and $b \in B\}$. We often write aRb for $(a, b) \in R$. For any binary relation R, we denote by $R^{-1} = \{(b, a)|aRb\}$ the inverse relation of R. When R is a binary relation from A to A, then R is said to be a binary relation on A. The set of all binary relations on A is denoted by Rel(A). In the following, we introduce some properties of binary relations. The binary relation R on A is (1) reflexive, if xRx for all $x \in A$; (2) symmetric, if xRy implies yRx for all $x, y \in A$; (3) antisymmetric, if xRy and yRx imply $x = y$ for all $x, y \in A$; (4) transitive, if xRy and yRz imply xRz for all $x, y, z \in A$. If a binary relation R is reflexive, symmetric and transitive, it is called an equivalence relation. We usually use E to denote an equivalence relation.

Let U be a nonempty finite set and E be an equivalence relation on U. The subset $[x]_E = \{y \mid xEy\}$ is the equivalence class containing x for every $x \in U$. Then E generates a partition $U/R = \{[x]_E|x \in U\}$, namely, a family of pairwise disjoint subsets whose union is the universe. For any $X \subseteq U$, the (Pawlak) lower approximation and upper approximation of X, denoted by $\underline{apr}(X)$ and $\overline{apr}(X)$ respectively, are defined by

$$\underline{apr}(X) = \bigcup\{[x]_E \mid x \in U, [x]_E \subseteq X\}$$
$$\overline{apr}(X) = \bigcup\{[x]_E \mid x \in U, [x]_E \cap X \neq \emptyset\}. \tag{Def 1}$$

If $\underline{apr}(X) = \overline{apr}(X)$, then X is called a definable set, otherwise X is called a (Pawlak) rough set. The set $BN_E(X) = \overline{apr}(X) - \underline{apr}(X)$ will be referred as the R-boundary region of X. The set of all definable sets is denoted by $\mathcal{D}(U/E)$, which can be obtained from U/E by adding $\emptyset$ and making it closed under set union. As we know, $\mathcal{D}(U/E)$ can be interpreted in different ways, such as the family of all open and closed sets in the topological space $(U, \mathcal{D}(U/E))$ or a Boolean sublattice of $(\mathcal{P}(U), \subseteq)$ (where $\mathcal{P}(U)$ is the powerset of U, i.e., the set of all subsets of U). Besides, we shall point later that $\mathcal{D}(U/E)$ is the family of all open and closed sets of a matroid.

2.2. Matroids

Let U be a finite set and $\mathcal{I}$ be a nonempty subset of $\mathcal{P}(U)$, then $(U, \mathcal{I})$ is a set system. A set system $(U, \mathcal{I})$ is called a matroid if the following conditions hold:

(I1) If $X \in \mathcal{I}$, and $Y \subseteq X$, then $Y \in \mathcal{I}$.
(I2) Let $X, Y \in \mathcal{I}$, and $|X| < |Y|$ (where $|X|$ denotes the cardinality of X), then there exists a set $Z \in \mathcal{I}$ such that $X \subset Z \subseteq X \cup Y$.

Let $M = (U, \mathcal{I})$ be a matroid. The members of $\mathcal{I}$ are the independent sets of M. A set in $\mathcal{I}$ is maximal in the sense of inclusion is called a base of the matroid M. A subset A of U is called dependent if $A \notin \mathcal{I}$. A minimal, in the sense of inclusion, dependent subset of U is called a circuit of the matroid M. For the family of all bases, all circuits, and all dependent sets