Information Sciences 253 (2013) 100–109

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Multi-adjoint relation equations: Definition, properties and solutions using concept lattices

Juan Carlos Díaz, Jesús Medina *

Department of Mathematics, University of Cádiz, Spain

ARTICLE INFO

Article history:
Received 12 March 2012
Received in revised form 17 June 2013
Accepted 28 July 2013
Available online 6 August 2013

Keywords:
Fuzzy relation equation
Galois connection
Property-oriented concept lattice

ABSTRACT

This paper generalizes fuzzy relation equations following the multi-adjoint philosophy. Moreover, the solutions of these general fuzzy relation equations and the concepts of a multi-adjoint property-oriented concept lattice are related, and several results are obtained from the theory of concept lattices.

As a consequence of this relevant relation, more properties about these general equations can be proven from the theory of concept lattices and the algorithms developed to compute concept lattices can be used to obtain solutions. Furthermore, an interesting application to fuzzy logic programming has been introduced, in which an important problem in this topic has been interpreted in terms of solving a multi-adjoint relation equation.

1. Introduction

Fuzzy relation equations, introduced by Sanchez [37], are associated with the composition of fuzzy relations and are used to investigate theoretical and applicational aspects of fuzzy set theory [14]. Many papers have investigated the capacity to solve (systems of) fuzzy relation equations, e.g. in [1,7,13,14,33,35].

On the other hand, multi-adjoint property-oriented concept lattices have been introduced in [23] as a generalization of property-oriented concept lattices [17,18] in a fuzzy environment, that embeds the theory given in [19,20]. These concept lattices provide a new point of view of rough set theory proposed by Pawlak in the 1980s, which is a formal tool for modeling and processing incomplete information in information systems [31].

This paper introduces a generalization of fuzzy relation equations in order to consider the flexibility provided by the multi-adjoint philosophy, generalizing the recent framework given in [2,5,7]. Moreover, an important fact which distinguishes this paper from the rest, is that multi-adjoint property-oriented concept lattices and multi-adjoint relation equations have been related in order to attain results that ensure the existence of solutions in these equations, extending those given in [16]. As a consequence, the theory given in [16,34] is generalized. These definitions and results are illustrated by an interesting application to fuzzy logic programming. An important problem in this topic has been interpreted as solving a multi-adjoint relation equation, which improves the readability and comprehension of this paper. Finally, the multi-adjoint relation equations are compared to other general approaches.

An important consequence of this relation is that it enables the properties given, e.g. in [3,4,6,19,20,25,27,36] to be applied to obtain many properties of these systems. Indeed, the algorithms presented, e.g. in [8,9,22], or further generalizations to compute the concept lattices in [23,24,27], can be applied to attain the solutions to these systems.

The plan of this paper is the following: a summary of multi-adjoint property-oriented concept analysis is introduced in Section 2. Section 3 introduces multi-adjoint relation equations together with a problem in fuzzy logic programming.

* Corresponding author. Tel.: +34 956012712.
E-mail addresses: juancarlos.diaz@uca.es (J.C. Díaz), jesus.medina@uca.es (J. Medina).

0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.07.024

Section 4 studies the existence of solutions of a multi-adjoint relation equation and several interesting results, from the viewpoint of concept lattice theory. The comparison to other approaches is introduced in Section 6. Lastly, the paper ends with several conclusions and prospects for future work.

2. Multi-adjoint property-oriented concept lattices

The basic operators in this environment are the adjoint triples [12], which are formed by three mappings: a possible non-commutativity conjunctor and two residuated implications, that satisfy the well-known adjoint property.

Definition 1. Let $(P_1,\leqslant_1)$, $(P_2,\leqslant_2)$, $(P_3,\leqslant_3)$ be posets and $\&\colon P_1\times P_2\to P_3$, $\swarrow\colon P_3\times P_2\to P_1$, $\nwarrow\colon P_3\times P_1\to P_2$ be mappings, then $(\&,\swarrow,\nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

$$x\leqslant_1 z\swarrow y \ \text{ iff } \ x\& y\leqslant_3 z \ \text{ iff } \ y\leqslant_2 z\nwarrow x \tag{1}$$

where $x\in P_1$, $y\in P_2$ and $z\in P_3$.

Equivalence (1) is called *adjoint property*. These operators are a straightforward generalization of t-norms and its residuated implication. Since a t-norm is commutative, in this case both implications coincide. In [28] more general examples of adjoint triples are given.

Example 1. Let $[0,1]_m$ be a regular partition of $[0,1]$ into m pieces, for example $[0,1]_2=\{0,0.5,1\}$ divides the unit interval into two pieces.

A discretization of a t-norm $\&\colon[0,1]\times[0,1]\to[0,1]$ is the operator $\&^*\colon[0,1]_n\times[0,1]_m\to[0,1]_k$, where $n,\ m,\ k\in\mathbb{N}$, and which is defined, for each $x\in[0,1]_n$ and $y\in[0,1]_m$, as:

$$x\&^*y=\frac{\lceil k\cdot(x\&y)\rceil}{k}$$

where $\lceil\_\rceil$ is the ceiling function.

For this operator, the corresponding residuated implications $\swarrow^*\colon[0,1]_k\times[0,1]_m\to[0,1]_n$ and $\nwarrow_*\colon[0,1]_k\times[0,1]_n\to[0,1]_m$ are defined as:

$$z\swarrow^*y=\frac{\lfloor n\cdot(z\leftarrow y)\rfloor}{n}\qquad z\nwarrow_*x=\frac{\lfloor m\cdot(z\leftarrow x)\rfloor}{m}$$

where $\lfloor\_\rfloor$ is the floor function and $\leftarrow$ is the residuated implication of the t-norm $\&$.

The triple $(\&^*,\swarrow^*,\nwarrow_*)$ is an adjoint triple, although the operator $\&^*$ could be neither commutative nor associative.

The basic structure, which allows the existence of several adjoint triples for a given triplet of lattices, is the multi-adjoint property-oriented frame.

Definition 2. Given two complete lattices $(L_1,\preceq_1)$ and $(L_2,\preceq_2)$, a poset $(P,\leqslant)$ and adjoint triples with respect to P, L_2, L_1, $(\&_i,\swarrow^i,\nwarrow_i)$, for all $i=1,\ldots,l$, a *multi-adjoint property-oriented frame* is the tuple $(L_1,L_2,P,\&_1,\ldots,\&_l)$.

The definition of context in this framework is analogous to the one given in [27].

Definition 3. Let $(L_1,L_2,P,\&_1,\ldots,\&_l)$ be a multi-adjoint property-oriented frame. A *context* is a tuple (A,B,R,σ), where A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A\times B\to P$ and $\sigma\colon B\to\{1,\ldots,l\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.

From now on, we will fix a multi-adjoint property-oriented frame and context, $(L_1,L_2,P,\&_1,\ldots,\&_l)$, (A,B,R,σ) and, to improve readability, we will write $\&_b$, $\nwarrow_b$ instead of $\&_{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

In this environment, the following mappings ${}^{\uparrow_\pi}\colon L_2^B\to L_1^A$ and ${}^{\downarrow^N}\colon L_1^A\to L_2^B$ are defined, for each $a\in A$, $b\in B$, as

$$g^{\uparrow_\pi}(a)=\sup\{R(a,b)\&_b g(b)\mid b\in B\}$$
$$f^{\downarrow^N}(b)=\inf\{f(a)\nwarrow_b R(a,b)\mid a\in A\}$$

The pair $({}^{\uparrow_\pi},{}^{\downarrow^N})$ is an isotone Galois connection [23,24], that is ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$ are order-preserving; and they satisfy that $f^{\downarrow^N\uparrow_\pi}\preceq_1 f$, for all $f\in L_1^A$, and that $g\preceq_2 g^{\uparrow_\pi\downarrow^N}$, for all $g\in L_2^B$.

A pair of fuzzy subsets $\langle g,f\rangle$, with $g\in L^B$, $f\in L^A$, such that $g^{\uparrow_\pi}=f$ and $f^{\downarrow^N}=g$, will be called *multi-adjoint property-oriented concept*. In that case, g is called the *extent* and f the *intent* of the concept. The set of all these concepts will be denoted by $\mathcal{M}_{\pi N}$ and, together with the ordering $\preceq$ defined by $\langle g_1,f_1\rangle\preceq\langle g_2,f_2\rangle$ iff $g_1\preceq_2 g_2$(or equivalently $f_1\preceq_1 f_2$), forms a complete lattice [23,24], $(\mathcal{M}_{\pi N},\preceq)$, which is called *multi-adjoint property-oriented concept lattice*.

A similar theory is developed if σ is defined on A, $\sigma\colon A\to\{1,\ldots,l\}$.