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# Two-dimensional discrete fuzzy numbers and applications[☆]

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ABSTRACT

In this paper, the concept of two-dimensional discrete fuzzy numbers is presented based on the idea of representation theorem of one-dimensional discrete fuzzy numbers. Then a sufficient condition is proposed, which makes a fuzzy set on $R^2$ become a two-dimensional discrete fuzzy number, and the representations of joint membership function and the two edge membership functions of two-dimensional discrete fuzzy number are given. Some special two-dimensional discrete fuzzy numbers are also defined. And then some weak orders in the two-dimensional discrete fuzzy number space are set up based on the mass centers and the degree of ambiguities of two-dimensional discrete fuzzy numbers, and their properties are also investigated. At last, some practical examples are included to demonstrate the effectiveness and potential of the theoretic results obtained.



## 1. Introduction

The concept of fuzzy set was first introduced in [36]. Then the concept of (continuous) fuzzy numbers was proposed in [10] to study the properties of probability functions. With the development of theories and applications of fuzzy numbers, see for example, [4,11,12,17,20,21,37], this concept becomes more and more important. Recently, there is still a lot of work about one-dimensional (continuous) fuzzy numbers. For example, in [28], Sanz and Fernández et al. improved the performance of fuzzy rule-based classification systems with interval-valued fuzzy sets and genetic amplitude tuning; In [3], Asady proposed a revision of distance minimization method for ranking of fuzzy numbers; In [24], Moreno-Garcia and Linares et al. proposed a method to construct trapezoidal fuzzy number approximations from raw discrete data; In [16], Coroianu and Gagolewski studied the problem of the nearest approximation of fuzzy numbers by piecewise linear one-knot fuzzy numbers. About multi-dimensional (continuous) fuzzy numbers, there is also a lot of work. For example, in [32], we proposed some fuzzy approximation relations on fuzzy n-cell number space, and gave their applications in classification; In [19], Hong considered the law of large numbers for T-related weighted fuzzy variables whose underlying spaces are $R^p$; In [2], Arotaritei and Ionescu introduced ařfuzzy Voronoiaś diagrams for fuzzy numbers of dimension two by extension of Voronoi diagrams for fuzzy numbers; In [31], we defined fuzzy ellipsoid numbers, and gave a method to construct such fuzzy numbers to represent uncertain or imprecise multi-channel digital information.

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The concept of one-dimensional discrete fuzzy numbers was given in [29]. In 2005, we obtained a representation theorem of discrete fuzzy numbers by using $r$-level sets, and investigated operations of discrete fuzzy numbers based on the representation theorem in [33]. Subsequently, Casasnovas and Riera studied some properties of discrete fuzzy numbers were in [5–9]. Recently, Riera and Torrens studied the aggregation functions defined on the set of all discrete fuzzy numbers whose supports are subsets of consecutive natural numbers, and put the obtained results into the aggregation of subjective evaluations in [25]. Riera and Torrens also discussed the residual implications on the set of discrete fuzzy numbers in [26], and proposed a method to construct aggregation functions on the set of discrete fuzzy numbers whose supports are sets of consecutive natural numbers from a couple of discrete aggregation functions in [27]. In addition, Massanet et al. gave a new linguistic computational model based on discrete fuzzy numbers for computing with words in [23]. Ariño and Pérez presented a procedure for obtaining polytopic λ-contractive sets for Takagi–Sugeno fuzzy systems in [1].

The discrete fuzzy numbers which are discussed in the above works are all one-dimensional discrete fuzzy numbers, which can be only used to express and deal with some uncertain single channel digital information. For multichannel digital information, one-dimensional discrete fuzzy numbers are no longer suitable. For example, although one-dimensional discrete fuzzy numbers can be used to express the pixel value in the center point of a window for a black and white image [33], they cannot be used for a color image (since any color can be composed of three basic colors 'red, yellow and blue', it can be only considered that using a three-dimensional discrete fuzzy number to express the pixel value of one point in a color image). Another example, as we evaluate an interviewee (denoted by $O$) looking for a job, we should not only evaluate his (or her) working ability, but also evaluate his (or her) moral quality. Owing to two evaluation factors, we should consider that using a two-dimensional discrete fuzzy number to complete the evaluation. In addition, since the objects considered by us in Example 5.1 are characterized with two evaluation factors, it is also suitable by using two-dimensional discrete fuzzy numbers to express the objects. Therefore, studying multi-dimensional discrete fuzzy numbers is more important and meaningful. Without loss of generality, we only study two-dimensional discrete fuzzy numbers in this paper. Then the first task is how strictly to define two-dimensional discrete fuzzy numbers. In [29], Voxman used the total order "≤" (or "≥") in $R$ giving the definition of one-dimensional discrete fuzzy numbers. But in $R^2$, there is no total order "≤" (or "≥"), so we cannot define two-dimensional discrete fuzzy numbers like the method in [29]. Of course, we should also pay attention to that two-dimensional discrete fuzzy numbers is different from type-2 fuzzy sets (theories and applications related to type-2 fuzzy sets can be referred to [13–15,18,22,34,35,38]) since a type-2 fuzzy set on $R$, in essence, can be understood as a mapping from $R$ into the set (space) which is the collection of all fuzzy sets on $R$. However a two-dimensional discrete fuzzy number should be a mapping from $R^2$ into closed interval $[0, 1]$. So we cannot also learn from the method defining type-2 fuzzy sets to define two-dimensional discrete fuzzy numbers. How a reasonable definition of two-dimensional discrete fuzzy numbers is given? The representation theorem (Theorem 3.1 in [33]) of one-dimensional discrete fuzzy numbers gives us an inspiration. Using the idea of the representation theorem of one-dimensional discrete fuzzy numbers, by introducing the concepts $X \leq\leq Y$ (or $Y \geq\geq X$) and $X \leq\geq Y$(or $Y \geq\leq X$) for $X, Y \in R^2$, we can obtain a method with originality to define two-dimensional discrete fuzzy numbers (Definition 3.1).

In this paper, firstly we give the definition of two-dimensional discrete fuzzy numbers. Then a sufficient condition with originality is obtained (Theorem 3.2), which makes a fuzzy set on $R^2$ become a two-dimensional discrete fuzzy number. A representation with a little of originality of joint membership function and the two edge membership functions of a two-dimensional discrete fuzzy number by each other are given (Theorems 3.2 and 3.5). Some special two-dimensional discrete fuzzy numbers are defined, one of which is called two-dimensional unite discrete fuzzy number. And then we set up some weak orders with a little of originality in the two-dimensional discrete fuzzy number space based on the mass centers and the degree of ambiguities of two-dimensional discrete fuzzy numbers (Definitions 4.2 and 4.5), and investigate their properties. Finally, we present some practical examples to show the proposed weak orders in the two-dimensional discrete fuzzy number space used to rank some objects which are characterized by two sets of uncertain digital information.

## 2. Preliminaries

Let $R^n$ be the $n$-dimensional Euclidean space. A fuzzy set of $R^n$ is a function $u : R^n \to [0, 1]$. For each fuzzy set $u$ of $R^n$, let $[u]^r = \{x \in R^n : u(x) \geq r\}$ for any $r \in (0, 1]$ be its $r$-level set (or $r$-cut). We denote the support of $u$ by $\mathrm{supp}(u)$, i.e., $\mathrm{supp}(u) = \{x \in R^n : u(x) > 0\}$. In addition, we denote the closure of $\mathrm{supp}(u)$ by $[u]^0$, i.e., $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$.

**Definition 2.1.** [29] A fuzzy set $u : R \to [0, 1]$ is called a discrete fuzzy number if the support of $u$ is finite, i.e., there exist $x_1, x_2, \ldots, x_n \in R$ with $x_1 < x_2 < \cdots < x_n$ such that $[u]^0 = \{x_1, x_2, \ldots, x_n\}$, and there exist natural numbers $s, t$ with $1 \leq s \leq t \leq n$ such that

(1) $u(x_i) = 1$ for any natural number $i$ with $s \leq i \leq t$;
(2) $u(x_i) \leq u(x_j)$ for any natural numbers $i, j$ with $1 \leq i \leq j \leq s$;
(3) $u(x_i) \geq u(x_j)$ for any natural numbers $i, j$ with $t \leq i \leq j \leq n$.

We denote the collection for all discrete fuzzy numbers by $D$. The following theorem gives a characterization of discrete fuzzy numbers.

**Theorem 2.2.** [33] *Let $u \in D$. Then the following statements* $(1) \sim (4)$ *hold:*

(1) $[u]^r$ is a nonempty finite subset of $R$ for any $r \in [0, 1]$;