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Optimality conditions for generalized differentiable interval-valued functions ☆

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A B S T R A C T

In this paper we study the optimal solutions set for a generalized differentiable interval-valued function. Necessary and sufficient optimality conditions are established for gH-differentiable functions. Convexity assumptions that are necessary or required to ensure the characterization of the optimal solutions are weaker or less strict than those presented in previous works. These convexity assumptions are the weakest to characterize the optimal solutions set. Known results for classical non interval-valued optimization are particular cases of the ones proved here.

1. Introduction

Data in many real-life engineering and economical problems are not exact. Under most conditions, decision information is usually uncertain, due to the increasing complexity of the environment and the vagueness of the inherent subjective nature of human thought; thus, crisp values are inadequate or insufficient to model real-life decision problems; it might not be flexible or convenient for decision-makers to exactly quantify their opinions with crisp numbers. A possible solution is to represent such membership degrees by interval values. Consequently, Interval Analysis was introduced as an attempt to handle interval (nonstatistical, nonprobabilistic) uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The interval-valued optimization problems may provide an alternative choice for considering the uncertainty into the optimization problems.

Numerous examples where the uncertainty in real problems is formulated using Interval Analysis as mathematic tool can be found in the literature. In [8] the imprecise data in Data Envelopment Analysis (DEA) is addressed. The term "imprecise data" reflects the situation where some of the input and output data are only known to lie within bounded intervals (interval numbers). A DEA model with interval data is proposed.

In [30] a simple optimization example formulated as an optimization problem with an interval objective function is considered.

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Inuiguchi and Kume [12], formulated and solved goal programming problems with interval coefficients in which the targets values were also assumed as closed intervals. In [21,22] the authors describe a Goal Programming procedure for modeling and solving land utilization planning problems and academic resource planning problems in university management system having interval-valued objective goals for optimal production of seasonal crops in agricultural system and considering as interval-valued goals to make a satisfactory decision regarding staff allocation for smooth functioning of the academic activities departments. In the problems model formulation, the interval-valued goals defined are converted into conventional goals.

The interval-valued optimization problems are closely related with inexact linear programming problems [25]. A stochastic robust optimization problem of residential micro-grid energy management is presented in [16]. Combined cooling, heating and electricity technology (CCHP) is introduced to satisfy various energy demands. Interval programming methods are exploited to gain interval robust solutions under different robustness levels which are feasible for uncertain data. The obtained results can help micro-grid managers minimizing the investment and operation cost with lower system failure risk when facing fluctuant energy market and uncertain technology parameters.

Inuiguchi and Sakawa [13] introduced the minimax regret solution concept. In [19], an interval-valued minimax regret analysis (IMRA) method is proposed for planning greenhouse gas (GHG) abatement under uncertainty. The IMRA method is a hybrid of interval-parameter programming and minimax regret analysis techniques. The developed method is applied to support long-term planning of GHG mitigation in an energy system under uncertainty. It can help decision makers to identify an optimal strategy that can facilitate reducing the worst regret level incurred under any outcome of the uncertain GHG-abatement targe.

We can find many articles related with fuzzy optimization application, see for instances [18,24]. Our method developed herein can be used as a mathematical programming problem series on alpha-levels. Interval analysis is a key first component of any fuzzy interval analysis in general and in fuzzy optimization in particular.

In all previous examples the real problem is modeled by an interval-valued optimization problem, that is adequately transformed into classic optimization models, or an approach is searched in order to find a solution. In this paper we prove necessary and sufficient optimality conditions that generalize the ones in classic differentiable optimization.

Many approaches to interval-valued optimization problems have been explored in considerable details, and it has been studied in several works [5,14,17]. In Classical Optimization Theory, the stationary point or critical point concept as the one that annuls the gradient of the function to optimize, is fundamental, both from the theoretical and practical point of view. In this work we tackle the interval-valued optimization theory in this same key aspect. The problem is that in order to define the stationary point concept properly in the environment of interval-valued functions we need a derivative concept which is both theoretically well founded and it is also applicable.

Hukuhara derivative for a set-valued mapping was first introduced by Hukuhara in [10]. In order to overcome some shortcomings of this approach, other types of derivatives for set-valued functions have been explored (see [2,7,11]). A derivative concept slightly more general than the previous concepts defined for interval-valued functions was presented in [26]. That new concept was defined using a generalization of the Hukuhara difference for compact convex sets (*gH*-difference). A detailed study of the *gH*-derivative properties and a characterization of *gH*-differentiable functions in terms of the differentiability of the real-valued functions giving endpoint of the interval were presented in [4]. Other properties of the *gH*-derivative was obtained in [3,1].

In order to study the interval-valued optimization problems, Wu [27–30], derived sufficient Karush–Kuhn–Tucker-type optimality conditions for multiobjective programming problems with interval-valued objective functions and Pareto optimal solution concepts are proposed considering two different orderings on the class of all closed intervals, and he discussed Wolfe-type duality theorems. But the H-differentiability and convexity of endpoint functions are assumed. In [15] the convexity assumption is relaxed supposing that H-differentiability is verified, and they obtain sufficient optimality conditions and Mond–Weir and Wolfe type duality relations for interval-valued programming problems. In these works the assumptions are very restrictive. For instance, $F(x) = Cx$, where C is any interval and x is a real number, is not convex being that F is a generalization of a linear function. Also, they do not provide necessary optimality conditions. So, we are going to introduce in this paper a more adequate generalized convexity concept to weaken the differentiability hypothesis, and we prove necessary optimality conditions.

The fundamental contributions of this paper are: we find necessary optimality conditions, that have not been studied in the literature up to now, and prove sufficient optimality conditions under convexity hypotheses that are weaker than those established in other previous works, and we also proof, besides, that they are as weak as possible. All this, considering that the functions are gH-differentiable, being this differentiability definition more general than others in the literature and therefore the optimality conditions obtained are applicable to a wider range of functions. All the classical optimization results (when you have non interval-valued functions) are particular instances of the ones presented here.

This paper is organized as follows: in Section 2 we introduce some basic properties and arithmetic for intervals and, a brief example as our study motivation. Using the generalized Hukuhara difference (*gH*-difference) and the limit concept for interval-valued functions, we consider the *gH*-derivative for an interval-valued function. In Section 3, we provide different minimum concepts for a interval-valued function defined on $\mathbb{R}^n$ and we derive necessary optimality conditions form the stationary point or critical point definition. In Section 4, a new generalized convexity concept is introduced and it is proved that this is the weaker hypothesis in order to prove sufficient optimality conditions. Numerical examples are given to illustrate theoretical developments.