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Generators of copulas and aggregation

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ABSTRACT

The most applied class of copulas for fitting purposes is undoubtedly the class of Archimedean copulas due to their representation by means of single functions of one variable, i.e., by means of additive generators or the corresponding pseudo-inverses. In this paper, we characterize aggregation functions preserving additive generators (pseudo-inverses of additive generators) of Archimedean copulas. As a by-product, we obtain an efficient method to construct new additive generators from some given ones.

1. Introduction

Since their proposal by Sklar [22], copulas became an important tool for the study and modelling problems dealing with random vectors. In this paper we will consider only bivariate random vectors, and hence only bivariate copulas. From the axiomatic point of view, copula is a function $C: [0,1]^2 \to [0,1]$ satisfying

(a) the boundary conditions $C(u,0) = C(0,v) = 0$ (C is grounded), $C(u,1) = u,\ C(1,v) = v$ (1 is neutral element of C)
(b) 2-increasing property $C(u,v) + C(u',v') - C(u,v') - C(u',v) \geqslant 0$ for all $u,v,u',v' \in [0,1],\ u \leqslant u',\ v \leqslant v'$.

From the statistical point of view, a copula $C: [0,1]^2 \to [0,1]$ is a function such that for any marginal distribution functions $F_X, F_Y: R \to [0,1]$ of random variables X and Y, the function $F_Z: R^2 \to [0,1]$ given by

$$F_Z(x,y) = C(F_X(x), F_Y(y)),$$

is a joint distribution function of some random vector $Z = (X,Y)$. Copula C describes here the dependence structure of the random vector Z. For more details we recommend [11,20].

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A special class of copulas is characterized by associativity, $C(C(u,v),w) = C(u,C(v,w))$ and diagonal inequality $C(u,u) < u$ (for all $u \in]0,1[$). This class appearing first in the study of probabilistic metric spaces [21], is called the class of Archimedean copulas, and due to Moynihan [19] we have the next important characterization result.

Theorem 1 (Moynihan [19]). *A function $C: [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous strictly decreasing convex function $f: [0,1] \to [0,\infty]$ satisfying $f(1) = 0$, called an additive generator, so that*

$$C(u,v) = g(f(u) + f(v)),$$

where $g: [0,\infty] \to [0,1]$ is given by $g(x) = f^{-1}(Min(f(0),x))$, and it is called a pseudo-inverse of f.

Note that pseudo-inverses were deeply discussed in [14], and for a decreasing non-constant function $h: [c,d] \to [a,b]$ the corresponding pseudo-inverse is given by

$$h^{(-1)}(x) = \sup\{t \in [a,b] | h(t) > x\},$$

with the convention $\sup \emptyset = a$. It is not difficult to check that, when considering the functions f, g from Theorem 1, then $g = f^{(-1)}$ and $f = g^{(-1)}$, i.e., the information contained in the additive generator f of an Archimedean copula C is the same as the information contained in its pseudo-inverse g, and that $C(u,v) = f^{(-1)}(f(u) + f(v)) = g(g^{(-1)}(u) + g^{(-1)}(v))$. In fact, in the literature both forms are used, the first one (based on an additive generator f) being preferred in probabilistic areas, while the second one (based on g) is more frequently used in the statistical literature.

We denote by $\mathcal{F}$ the class of all additive generators f, i.e., of all functions $f: [0,1] \to [0,\infty]$ which are continuous, strictly decreasing, convex and satisfying $f(1) = 0$. Observe that if $f \in \mathcal{F}$ then also $cf \in \mathcal{F}$ for any positive constant $c \in]0,\infty[$, and that if an Archimedean copula C is generated by two additive generators f_1 and f_2, then necessarily $f_1 = cf_2$ for some $c \in]0,\infty[$.

We denote by $\mathcal{G}$ the class of all pseudo-inverses g of additive generators $f \in \mathcal{F}$.

Lemma 1. *A function $g: [0,\infty] \to [0,1]$ belongs to $\mathcal{G}$ if and only if it is continuous, convex, $g(0) = 1$, and there is a constant $a \in]0,\infty]$ such that g is strictly decreasing on $[0,a]$, and $g(x) = 0$ for all $x \in [a,\infty]$.*

Proof. Suppose $g \in \mathcal{G}$, i.e., there is $f \in \mathcal{F}$ such that $g = f^{(-1)}$. Then clearly $g(0) = 1$, and due to results from [14], g is continuous and decreasing. Moreover, denoting $a = f(0)$, g is vanishing on $[a,\infty]$. Then the convexity of g on $[0,\infty]$ is guaranteed by the convexity of g on $[0,a]$. Finally, the function $h: [0,a] \to [0,1]$, $h(x) = g(x)$, is the inverse of f, $h = f^{-1}$ and thus strictly decreasing and convex, concluding the necessity part of this lemma. To see the sufficiency, it is enough to consider $f = h^{-1}$, where $h: [0,a] \to [0,1]$ is given as above, $h(x) = g(x)$. Then evidently $f \in \mathcal{F}$ and $g = f^{(-1)}$, i.e., $g \in \mathcal{G}$. $\square$

Observe that Lemma 1 allows to consider the class $\mathcal{G}$ independently of the class $\mathcal{F}$.

The aim of this paper is to study construction methods for additive generators (or their pseudo-inverses) of Archimedean copulas from some a priori given additive generators (pseudo-inverses) by means of aggregation functions. In the next section, we characterize aggregation functions preserving the class $\mathcal{F}$ of all additive generators of Archimedean copulas. Section 3 brings a characterization of aggregation functions preserving the class $\mathcal{G}$ of all pseudo-inverses of additive generators of Archimedean copulas. Finally, some concluding remarks are given.

2. Aggregation of additive generators of Archimedean copulas

Note, first of all, that the class $\mathcal{F}$ is convex, i.e., for any $f_1,\ldots,f_n \in \mathcal{F}$ and $c_1,\ldots,c_n \in [0,1]$, $\sum_{i=1}^n c_i = 1$, also $f = \sum_{i=1}^n c_i f_i \in \mathcal{F}$. Due to the already mentioned fact that any positive multiple cf of an additive generator $f \in \mathcal{F}$ is again an additive generator, $cf \in \mathcal{F}$, we see that one can relax the constraint $\sum_{i=1}^n c_i = 1$ into $\sum_{i=1}^n c_i > 0$, i.e., any non-trivial non-negative linear combination of additive generators from $\mathcal{F}$ is again an element of $\mathcal{F}$. For more details and examples see [1].

Recall that, for $n \in \{2,3,\ldots\}$, an aggregation function $A: [a,b]^n \to [a,b]$ is characterized by the increasing monotonicity in each coordinate and by boundary conditions $A(a,\ldots,a) = a$ and $A(b,\ldots,b) = b$. For more details we recommend [5,8,3], see also [9,10].

To aggregate additive generators $f_1,\ldots,f_n \in \mathcal{F}$ into an additive generator $f \in \mathcal{F}$, $f = A(f_1,\ldots,f_n)$, i.e., for all $x \in [0,1]$, $f(x) = A(f_1(x),\ldots,f_n(x))$, obviously one should consider the interval $[a,b] = [0,\infty]$ for inputs/output domain of considered values, i.e., we look for appropriate aggregation functions $A: [0,\infty]^n \to [0,\infty]$. It is immediate that due to the boundary condition for additive generators and agg. functions, $f(1) = A(f_1(1),\ldots,f_n(1)) = A(0,\ldots,0) = 0$, independently of A and $f_1,\ldots,f_n \in \mathcal{F}$. To ensure the continuity of f, A should be continuous. Similarly, to ensure the strict monotonicity of f, A should be jointly strictly increasing, i.e., $A(\mathbf{x}) < A(\mathbf{y})$ whenever $\mathbf{x},\mathbf{y} \in [0,\infty]^n$ and $x_i < y_i, i = 1,\ldots,n$. To ensure $f \in \mathcal{F}$ we have to ensure the convexity of f.

The next theorem gives a complete characterization of aggregation functions preserving the class $\mathcal{F}$ of all additive generators of copulas.