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Fuzzy valued probability ☆

Mila Stojaković [a,*], Ljiljana Gajić [b]

[a] University of Novi Sad, Faculty of Technical Sciences, Department of Mathematics, 21000 Novi Sad, Serbia
[b] University of Novi Sad, Faculty of Science, Department of Mathematics, 21000 Novi Sad, Serbia

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ABSTRACT

In this paper the main result is introduction of fuzzy valued probability restricted with respect to probability selectors. In order to treat a highly imprecise probability system, the fuzzy valued probability is defined over the measurable space as a fuzzy valued set function which is not additive in classical sense but it is additive with respect to restricted arithmetics. Some properties and the connection with fuzzy valued measure are discussed. Also, an application in Markov decision process with imprecise information is presented.

1. Introduction

Probability theory is usually used to treat uncertain system but in inference problems where the relevant information is scarce, vague or conflicting, incomplete data set delivers an imprecise assessment of the probability of an event. Mathematical models which measure chance or uncertainty without sharp numerical probabilities are covered by general name – imprecise probability. It includes both qualitative (comparative probability, partial preference orderings, etc.) and quantitative modes (interval probabilities, possibility theory, belief functions, upper and lower previsions, upper and lower probabilities, etc.). To deal with problems in which imprecision due to partial information play important role, probability theory and fuzzy logic are the main components of numerous methodologies. Imprecise assessment of the probability of an event can be expressed by a fuzzy set defined on $[0,1]$ instead by a number. It provides to probability theory an extra dimension of uncertainty. The fuzzy logic expresses the imprecision in probabilities explicitly in order to signal the appropriate level of confidence ascribed to them.

One of most attractive way to formalize a highly imprecise probabilistic system could be by using the theory of fuzzy valued random variables or using the theory of fuzzy valued probability. Some, among the many of the studies concerning this issue, can be find in [3,5,7–9,11,12,15–17]. A common component in all of these generalized methods, is that they focus on realistic reflection of available information and preferences, and as such support informative decisions.

In [12] a notion of fuzzy valued probability (derived from fuzzy valued measure) is defined and its properties discussed. The new concept (using probability selectors) of imprecise probability – named by the same name "fuzzy valued probability" – is introduced in this paper. Example 5 in Section 5 shows that the definition given in this paper is a genuine generalization of the concept from [12]. Fuzzy valued probability is defined over the measurable space with values in the set of fuzzy sets of

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* Corresponding author.
E-mail addresses: stojakovic@sbb.rs (M. Stojaković), gajic@dmi.uns.ac.rs (L. Gajić).

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unit interval. The method of restricted set arithmetics [6] is used to treat the probabilities which are fuzzy valued, but in spite of that the sum of all the individual probabilities is 1. That kind of fuzzy valued probability still has some nice properties – it is normed and $*$additive, where $*$additivity is the additivity with respect to addition in restricted arithmetics. The advantage and the motivation of using restricted $*$addition instead of the traditional addition lies in the advantage of preserving crucial property of the probability – property that the sum of probabilities of complemented events is 1. With classical addition of fuzzy sets this property fails (see Example 1). Unfortunately, from the computational viewpoint, restricted addition is not as straightforward as the classical one. In applications of our model and other similar models of uncertain probabilities, most of computations usually involve rather complicated mathematical procedures such as linear and nonlinear programming and other methods of optimization theory.

One can consider our concept of uncertain probability as the extension and generalization of the classical model of probability theory. This model is suitable to generalize the single, interval valued model and some mathematical models where weaker information states are mathematically formalized in various ways.

It turns out that fuzzy valued probability is strongly connected with fuzzy valued measure. In this paper the connection in both ways (fuzzy valued probability $\leftrightarrow$ fuzzy valued measure) is investigated. Since there is no any assumption about convexity, this theory can be used to model and analyse probabilistic systems where the values of probability are highly imprecise but discrete. Some basic properties of fuzzy valued probability are given. Random variable in fuzzy probability space is introduced and mathematical expectation restricted with respect to probability selectors of related fuzzy probability is defined. This model is applied to Markov decision process operating indefinitely with deterministic stationary policy. It provides a powerful tool for optimizing the performance of stochastic process that can be modeled as a discrete time Markov chain under uncertain transition probabilities. The general motivation for use of fuzzy probabilities is that the more evidence on which a probability estimate is based, the more confidence a decision-maker can have in it.

2. Preliminaries

First, for the convenience of the reader, we give a list of symbols used in this paper:

$\mathbb{R}, \mathbb{R}^+$	set of reals, set of non negative reals
$\mathbb{N}$	set of natural numbers
$(\Omega, \mathcal{A})$	measurable space where $\mathcal{A}$ is a σ-algebra on Ω;
A'	complement of $A, A' = \Omega \backslash A$
$\mathcal{K}_{(f)(k)(c)}(\mathbb{R})$	set of nonempty (closed), (compact), (convex) subsets of $\mathbb{R}$
$\mathcal{F}_{(f)(k)(c)}(M)$	set of fuzzy sets on M with nonempty (closed), (compact), (convex) levels
h	Hausdorff metric on $\mathcal{K}_f(\mathbb{R})$
$\lvert A\rvert, A \subset \mathbb{R}$	$\lvert A\rvert = h(A, \{0\}) = \sup_{x \in A}\lvert x\rvert$
$clA, \overline{A}$	closure of the set A
$coA, \bar{c}oA$	convex hull, convex closure of the set A
I_A	characteristic (indicator) function of the set A
u_α	α-level of fuzzy set u
α	in index always denotes α-level of fuzzy set
$m, M, \mathcal{M}$	measure, set valued measure, fuzzy valued measure
$p, P, \mathcal{P}$	probability, set valued probability, fuzzy valued probability
$\lvert M\rvert, \lvert\mathcal{M}\rvert$	variation of M, variation of $\mathcal{M}$
$\lvert P\rvert, \lvert\mathcal{P}\rvert$	variation of P, variation of $\mathcal{P}$
S_M	set of measure selectors of set valued measure M
(a, b, c)	triangular fuzzy number, $a, b, c \in \mathbb{R}$

Set valued measure (see [4,2]) is a natural generalization of the single valued measure.

Definition 2.1. Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ a σ-algebra of measurable subsets of the set Ω. If $M : \mathcal{A} \to \mathcal{K}(\mathbb{R})$ is a mapping such that for every sequence $\{A_i\}_{i \in N}$ of pairwise disjoint elements of $\mathcal{A}$ the next equality is satisfied $M(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} M(A_i)$, where

$$\sum_{i=1}^{\infty} M(A_i) = \left\{ x \in \mathbb{R}, x_i \in M(A_i) : \; x = \sum_{i=1}^{\infty} x_i (uncond.conv.) \right\},$$

and $M(\emptyset) = \{0\}$, then M is a set valued measure.

By $\lvert M\rvert : \mathcal{A} \to \mathbb{R}^+$ we denote the single valued positive measure (called variation) defined by $\lvert M\rvert(A) = \sup \sum_{i=1}^{n} \lvert M(A_i)\rvert, A \in \mathcal{A}$, where supremum is taken over all finite measurable partition $\{A_i\}_{i=1}^{n}$ of A. If M is a positive set valued measures, then $\lvert M\rvert : \mathcal{A} \to \mathbb{R}^+, \lvert M\rvert(A) = \lvert M(A)\rvert$. A set valued measure M is of bounded variation iff $\lvert M\rvert(\Omega) < \infty$. Set valued measure M is σ- finite if there exists a countable measurable partition $\{A_k\}_{k \in \mathbb{N}}$ of Ω such that for every $k \in \mathbb{N}, \lvert M\rvert(A_k) < \infty$.