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Aggregation on Boolean multi-polar space: Knowledge-based vs. category-based ordering

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ABSTRACT

Three types of aggregation operators defined on a Boolean multi-polar space $\{0, 1, \ldots, m\}$ are compared. Three different monotonicity conditions that yield different types of aggregation operators are discussed and the structure of these aggregation operators on $\{0, 1, \ldots, m\}$ with additional properties of associativity and commutativity is studied for any $m \in \mathbb{N}$. The commutative associative Boolean multi-polar aggregation operators are shown to coincide with Boolean multi-polar uninorms. The structure of commutative associative SL aggregation operators is shown to be dependent on the structure of the class of Abelian idempotent semigroups based on $m + 1$ elements with an annihilator. We show that each commutative associative SL aggregation operator with annihilator 0 corresponds to a lower semi-lattice with bottom element 0. For $m > 2$ the class of commutative associative C aggregation operators is proved to be equal to the intersection of the class of commutative associative Boolean multi-polar aggregation operators and the class of commutative associative SL aggregation operators. Examples of the three types of commutative associative aggregation operators are shown and discussed for $m = 1, 2, 3$.

1. Introduction

Multi-polar aggregation operators were introduced in [12] (see also [13–17]) for aggregation of rule outputs in fuzzy rule-based classification systems. The output of such a fuzzy classification rule is expressed by a pair (k, x), where k is the consequent class of the rule, i.e., one of the m predefined classes, $k \in K^m$ for $K^m = \{1, \ldots, m\}$ and $x \in [0, 1]$ is the absolute value of the rule output that can be represented as a confidence that the pattern belongs to the class k implied by the given rule. As the classification system is based on a collection of several fuzzy classification rules the pairs $(k_1, x_1), (k_2, x_2), \ldots$, where (k_i, x_i) corresponds to the ith rule, should be aggregated in order to achieve the final classification for a given pattern. In [18] it was shown that concepts related to multi-polar aggregation can be found also in game theory, election methods, multi-criteria decision making and others.

Multi-polar aggregation extends both the unipolar and the bipolar aggregation [4,7–10]. Similarly as in the case of bipolarity (see [9]), where we distinguish two possible models: the bipolar univariate model and the unipolar bivariate model, also in the case of multi-polarity we distinguish multi-polar univariate model (simple multi-polarity) and the unipolar multi-variate model (extended multi-polarity), see [14,18]. Therefore we distinguish two possible input spaces for multi-polarity: the multi-polar space and the extended multi-polar space.

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Similarly as fuzzy logic extends the classical Boolean logic, aggregation operators defined on $[0,1]$ extend (Boolean) aggregation operators defined on $\{0,1\}$. Knowledge of Boolean aggregation operators is fundamental for further exploration of aggregation operators on $[0,1]$. Indeed, for example, triangular norms (conorms) were defined as extensions of the Boolean AND (OR) operator. Similarly, in the case of uninorms on $[0,1]$, the aggregation on $\{0,1\}^2$ plays an important role and the fact whether $U(0,1)=0$ or $U(0,1)=1$ determines whether the uninorm U is conjunctive or disjunctive. That is the reason why we would like to explore multi-polar aggregation operators defined on $K^m \times \{0,1\}$ and we will call such operators Boolean multi-polar aggregation operators. In other words, we would like to study aggregation defined on $\{0,1,\ldots,m\}$.

The notion of capacities related to the interval $[0,1]$ is closely linked to the aggregation on $[0,1]$. On one hand, capacities can be taken as a restriction of the aggregation to the grid points of the unit hypercube, on the other hand, using the Choquet integral [5] the values of a capacity can be interpolated into the aggregation on the whole unit hypercube $[0,1]^n$. The unipolar capacities were extended to the bipolar framework in [7–9,11] to the three possible concepts: bi-capacities (bipolar univariate model), generalized bipolar capacities (unipolar bivariate model), and bipolar capacities (the interstep between the previous two concepts). In [7,8] the authors studied also the Möbius transform for bi-capacities and the Choquet integral with respect to a bi-capacity. Möbius transform for bi-capacities due to Grabish and Lebreuche is defined in an asymmetric manner. Using a slightly different ordering Fujimoto defined the symmetric version of the Möbius transform for bi-capacities in [6]. This shows that a different ordering on the input space can yield a different concepts related to aggregation.

Inspired by the Belnap's four-valued logic [2] we can assume several orders on a set $\{0,1,\ldots,m\}$. In the case of Belnap's four-valued logic (see Fig. 1) the truth values have a structure of a bilattice with two possible orderings. The ordering $\leqslant_t$ represents the truth ordering and the ordering $\leqslant_k$ represents the knowledge ordering, where $A \leqslant_k B$ if it is more known about the truth or falsity of the statement B than of the statement A. Similarly, on a set $\{0,1,\ldots,m\}$ we can have a knowledge ordering, which shows our knowledge about the classification of the input. Beside this, we have m category orderings.

The aim of this paper is to compare three types of aggregation operators on $\{0,1,\ldots,m\}$: aggregation operators which are monotone with respect to the knowledge ordering, aggregation operators that are monotone with respect to all m category orderings, and Boolean multi-polar aggregation operators. Note that in the bipolar case the bipolar ordering and the two category orderings coincide. Thus the two possible orderings yield two different Möbius transforms for bi-capacities mentioned above.

In this paper we suppose that the order of inputs is not important and therefore all investigated aggregation operators are commutative. Moreover, we will focus only on idempotent aggregation operators as it is natural to require that inputs which are all from the same category k aggregate to the category k.

In Section 2 we will recall all necessary basic notions and define the three types of aggregation operators on $\{0,1,\ldots,m\}$. We will discuss the structure of the three types of aggregation operators on $\{0,1,\ldots,m\}$ with additional property of associativity for any $m \in \mathbb{N}$ (Section 3). In Section 4 we will show the three types of associative aggregation operators for $m=1,2,3$. Finally, we give our conclusions in Section 5. The study of another important case when the aggregation operators are not associative is left for the future work.

2. Basic notions

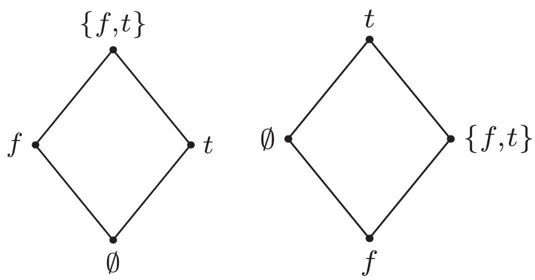
We begin with the definition of a (standard) aggregation operator on $[0,1]$.

Definition 1. A mapping $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \longrightarrow [0,1]$ is called an *aggregation operator* if

(A1) A is non-decreasing, i.e., for any $n \in \mathbb{N}, \mathbf{x}, \mathbf{y} \in [0,1]^n, \mathbf{x} \leqslant \mathbf{y}$ it holds $A(\mathbf{x}) \leqslant A(\mathbf{y})$.

(A2) $0,1$ are idempotent elements of A, i.e., for any $n \in \mathbb{N}$ there is $A\left(\underbrace{0,\ldots,0}_{n-\text{times}}\right) = 0$ and $A\left(\underbrace{1,\ldots,1}_{n-\text{times}}\right) = 1$.

(A3) for $n=1, A(x)=x$ for all $x \in [0,1]$.



(a) ordering $\leq_k$ (b) ordering $\leq_t$

Fig. 1. Belnap's four valued logic.