Information Sciences 295 (2015) 347–357

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Probabilistic-valued decomposable set functions with respect to triangle functions

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ARTICLE INFO

Article history:
Received 9 August 2013
Received in revised form 3 June 2014
Accepted 17 September 2014
Available online 13 October 2014

Mathematics Subject Classification (2010):
Primary 54E70
Secondary 60A10, 60B05

Keywords:
Probabilistic metric space
Decomposable measure
Triangle function
Triangular norm
Aggregation function

ABSTRACT

Real world applications often require dealing with the situations in which the exact numerical values of the (sub)measure of a set may not be provided, but at least some probabilistic assignment still could be done. Also, several concepts in uncertainty processing are linked to the processing of distribution functions. In the framework of generalized measure theory we introduce the probabilistic-valued decomposable set functions which are related to triangle functions as natural candidates for the "addition" in an appropriate probabilistic metric space. Several set functions, among them the classical (sub)measures, previously defined τ_T-submeasures, $\tau_{L,A}$-submeasures as well as recently introduced Shen's (sub)measures are described and investigated in a unified way. Basic properties and characterizations of τ-decomposable (sub)measures are also studied and numerous extensions of results from the above mentioned papers are provided.

1. Introduction

Consider a grant agency providing a financial support for research in some area. From the set of all grant applications only the "successful" ones (depending on some internal rules of agency) will receive certain amount of money. So, we have only a probabilistic information about the measure of the set of "successful" grant applications. Of course, the knowledge of this information depends on many different aspects: the total budget of money to be divided, the internal rules of the agency, the quality of reviewers (if any), etc. Further examples are provided by lotteries, or guessing results when we have a probabilistic information about the (counting) measure of possibilities to win the prize. A closely related concept can be found in Moore's interval mathematics [16], where the use of intervals in data processing is due to measurement inaccuracy and due to rounding. Here intervals can be considered in distribution function form linked to random variables uniformly distributed over the relevant intervals. These model examples resemble the original idea of Menger of probabilistic metric spaces. In his work [15], Menger proposed to replace a positive number by a distance distribution function. This fact was motivated by

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http://dx.doi.org/10.1016/j.ins.2014.09.047
0020-0255/

thinking of situations where the exact distance between two objects may not be provided, but some probability assignment is still possible. Thus, the importance/diameter/measure of a set might be represented by a distance distribution function. Recently, probabilistic approaches were successfully applied to modeling uncertain logical arguments [10], to approximations of incomplete data [6], to inference rules playing an important role in non-monotonic reasoning [4], or to cluster structure ensemble [25].

In the paper by Hutník and Mesiar [11] the notion of *τ_T-submeasure* was defined as a certain (non-additive) set function γ on a ring Σ of subsets of a non-empty set Ω, taking values in the set Δ^+ of distribution functions of non-negative random variables satisfying $\gamma_\emptyset = \varepsilon_0$, the "antimonotonicity" property $\gamma_E \geqslant \gamma_F$ whenever $E, F \in \Sigma$ with $E \subseteq F$, and "subadditivity" property of the form

$$\gamma_{E\cup F}(x+y) \geqslant T(\gamma_E(x), \gamma_F(y)), \quad E, F \in \Sigma, x, y > 0, \tag{1}$$

with T being a left-continuous t-norm. Here, ε_0 is the distribution function of Dirac random variable concentrated at point 0. As is shown in [11], such τ_T-submeasures can be seen as fuzzy number-valued submeasures. In this case the value γ_E may be seen as a non-negative LT-fuzzy number, see [3], where $\tau_T(\gamma_E, \gamma_F)$ corresponds to the T-sum of fuzzy numbers γ_E and γ_F. Also, each τ_M-submeasure γ with the minimum t-norm $M(x, y) = \min\{x, y\}$ can be represented by means of a non-decreasing system $(\eta_\alpha)_{\alpha\in[0,1]}$ of numerical submeasures as follows

$$\gamma_E(x) = \sup\{\alpha \in [0, 1];\ \eta_\alpha(E) \leqslant x\}, \quad E \in \Sigma.$$

This representation resembles the horizontal representation $(S_\alpha)_{\alpha\in[0,1]}$ of a fuzzy subset S.

The study of probabilistic-valued set functions continued in papers [8,9], where a more general concept has been used. In fact, in [9] a generalization of τ_T-submeasures was suggested involving suitable operations L, which replace the standard addition + on $\overline{\mathbb{R}}_+$. On the other hand, since t-norms are rather special operations on the unit interval $[0, 1]$, the paper [8] deals with a number of possible generalizations based on aggregation functions in general, studying certain properties of the corresponding probabilistic (sub)measures and their (sub)measure spaces.

Recently, in [23] Shen has defined and has studied a class of probabilistic (sub)measures which does not seem to fit into the concept of the above mentioned results of Hutník and Mesiar. In fact, Shen's definition of a probabilistic-valued $\top$-decomposable supmeasure[3] $\mathfrak{M} : \Sigma \to \Delta^+$ with the "subadditivity" property

$$\mathfrak{M}_{E\cup F}(t) \geqslant \top(\mathfrak{M}_E(t), \mathfrak{M}_F(t)), \quad E, F \in \Sigma, t > 0, \tag{2}$$

with $\top$ being a t-norm, corresponds to the notion of $\tau_{\max,T}$-submeasure defined in [9], however deeper contextual understanding was still unclear from that paper. Thus, in this paper we provide an even deeper insight into all the mentioned notions which are special cases of a *probabilistic-valued set function with respect to a triangle function*. Recall that a triangle function τ is a binary operation on Δ^+ such that the triple $(\Delta^+, \tau, \leqslant)$ forms a commutative, partially ordered semigroup with neutral element ε_0.

More precisely, the notion of τ_T-submeasure is related to the triangle function $\tau = \tau_T$ given by

$$\tau_T(G, H)(x) = \sup_{u+v=x} T(G(u), H(v)), \quad G, H \in \Delta^+ \tag{3}$$

with T being a left-continuous t-norm. Thus, the "subadditivity" property (1) resembles the "probabilistic analog" of the triangle inequality in the Menger probabilistic metric space (under T), see [22]. Moreover, $\tau_{L,T}$-submeasures defined in [9, Definition 2.3] are related to the (triangle) function

$$\tau_{L,T}(G, H)(x) = \sup_{L(u,v)=x} T(G(u), H(v)), \quad G, H \in \Delta^+, \tag{4}$$

with a suitable operation L on $\overline{\mathbb{R}}_+$. What is more, Shen's considerations are related to the pointwisely defined (triangle) function

$$\tau_\top(G, H)(t) = \top(G(t), H(t)), \quad G, H \in \Delta^+,$$

and the "subadditivity" property (2) is related to the triangle inequality of the corresponding probabilistic metric space. So, we can see that triangle functions are the main ingredients connecting all the mentioned notions of (sub)measure. Thus, considering a general triangle function τ on Δ^+, we define and study certain properties of τ-decomposable (sub)measures on a ring Σ of subsets of $\Omega \neq \emptyset$ in this general setting.

In the next section, the short overview of basic notions and definitions is given. In Section 3, we introduce the basic object of our study: a τ-decomposable set function with values in distance distribution functions and provide a number of concrete examples. Several properties of such set functions are then investigated in Section 4 and the results related to the probabilistic Hausdorff distance are provided in Section 6 generalizing the recent results of Shen [23].

[3] *supmeasure* in the terminology of Shen corresponds to *submeasure* in our terminology, see [23, Definition 4.1(v)].