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Some conditional vertex connectivities of complete-transposition graphs

Guoliang Wang, Haizhong Shi *, Feifei Hou, Yalan Bai

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730070, China

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ABSTRACT

A subset $F \subset V(G)$ is called an R^k-vertex-cut if $G - F$ is disconnected and each vertex $u \in V(G) - F$ has at least k neighbors in $G - F$. The cardinality of a minimum R^k-vertex-cut is the R^k-vertex-connectivity of G and is denoted by $\kappa^k(G)$. The conditional connectivity is a measure to study the structure of networks beyond connectivity. Hypercubes form the basic classes of interconnection networks. Complete transposition graphs were introduced to be competitive models of hypercubes. In this paper, we determine the numbers κ^1 and κ^2 for complete-transposition graphs, $\kappa^1(CT_n) = n(n-1) - 2$, $\kappa^2(CT_4) = 16$ and $\kappa^2(CT_n) = 2n(n-1) - 10$ for $n \geqslant 5$.

1. Introduction

Let $G = (V, E)$ be a finite graph without loops and parallel edges. We follow [1,2,12] for terminology not given here.

It is well known that the underlying topology of an interconnection network can be modeled by a graph G, and the connectivity $\kappa(G)$ of G is an important measure for fault tolerance of the network. In general, the larger $\kappa(G)$, the more reliable the network. However, it underestimates the resilience of the network since it is the worst case that measures the probability of whose occurrence is very small (see [5,9] for a detailed explanation for the shortcomings of using $\kappa(G)$ to measure the network reliability). To overcome such shortcomings, Harary [6] introduced the concept of conditional connectivity by placing some requirements on the components of $G - F$. The R^k-vertex-connectivity follows this trend.

For a simple connected graph $G = (V, E)$, a subset $F \subset V(G)$ is called an R^k-vertex-set of G if each vertex $u \in V(G) - F$ has at least k neighbors in $G - F$. An R^k-vertex-cut of a connected graph G is a R^k-vertex-set F such that $G - F$ is disconnected. In network applications, vertices in F are considered faulty vertices while those in $V(G) - F$ are good vertices. The R^k-vertex-connectivity of G, denoted by $\kappa^k(G)$, is the cardinality of a minimum R^k-vertex-cut of G. The idea behind this concept lies in measuring fault tolerance of networks. Notice that the probability that all failures concentrate around a vertex is often small. For example, an n-dimensional hypercube Q_n has 2^n vertices with vertex connectivity n. It is known that every minimum vertex cut of Q_n is the set of neighbors of some vertex. Suppose n vertices fail in Q_n. Then the probability that these n vertices from a vertex cut is $2^n / \binom{2^n}{n}$, which is very small when n becomes lager. In the definition of R^k-vertex-set, the requirement that each good vertex has at least k good neighbors takes the above resilience into consideration.

* Corresponding author.
E-mail addresses: wgl20071014210@163.com (G. Wang), haizhong.shi@163.com (H. Shi).

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The conditional connectivity is a measure to study the structure of networks beyond connectivity. Another approach is to consider the structure of these networks after it is disconnected by deleting vertices. According to the structure, we mean how badly is the graph disconnected. If the resulting graph has one big component plus several small ones, then the core is still intact in the sense. Such a structural result is useful in the study of other connectivities. No polynomial time algorithm is known for the computation of $\kappa^k(G)$. However, it is known for some particular classes of graphs. In [4], Cheng et al. determined R^1 and R^2 for the class of Cayley graphs by 2-trees. In [5], Esfahanian proved that $\kappa^1(Q_n) = 2n - 2$, where Q_n is the n-dimensional cube. Later, Latifi et al. [9] and Oh and Choi [11] determined $\kappa^k(Q_n) = (n-2)2^k$, for $1 \leqslant k \leqslant n$. In [7], Hu and Yang proved that $\kappa^1(S_n) = 2n - 4$, where S_n is the n-dimensional star graph. κ^2 is determined for Cayley graphs generated by transposition trees [17]. In addition, Zhang et al. [19] given exact value of $\kappa^2(AG_n)$, where AG_n is the n-dimensional alternating group graph.

Hypercubes form the basic classes of interconnection networks. The class of star graphs and the class of complete transposition graphs were introduced to be competitive models of hypercubes. Indeed, subsequent research shows that they are superior to hypercubes in many ways. In [16], R^2 is determined for the star graphs. In this paper, we determine κ^1 and κ^2 for the complete-transposition graphs.

2. Preliminaries

A graph G is called regular if the degree of every vertex is the same. If $W \subset V$ is a set of vertices, then $G - W$ will denote the graph obtained from G by deleting the vertices in W, that is, the graph spanned by the vertex set $V \setminus W$. A set of edges is independent if no two of them are incident to the same vertex. Let Γ be a finite group, and let Δ be a set of elements of Γ such that the identity of the group does not belong to Δ. The Cayley graph $\Gamma(\Delta)$ is the directed graph with vertex set Γ with an arc directed from v to u if and only if there is an $s \in \Delta$ such that $u = vs$. The Cayley graph $\Gamma(\Delta)$ is strongly connected if and only if Δ generates Γ. If whenever $u \in \Delta$, we also have its inverse $u^{-1} \in \Delta$, then for every arc, the reverse arc is also in the graph. So we can treat this Cayley graph as an undirected graph by replacing each pair of arcs by an edge.

In this work, we consider Cayley graph Cay(Sym(n), $\mathcal{T}$), where Sym(n) is the symmetric group on $\{1, 2, \ldots, n\}$ and $\mathcal{T}$ is a set of transposition of Sym(n). Let $G(\mathcal{T})$ be the graph on n vertices $\{1, 2, \ldots, n\}$ such that there is an edge ij in $G(\mathcal{T})$ if and only if transposition $(ij) \in \mathcal{T}$ [13]. The graph $G(\mathcal{T})$ is called the transposition generating graph of Cay(Sym(n), $\mathcal{T}$). If $G(\mathcal{T})$ is a tree, Cay(Sym(n), $\mathcal{T}$) is denoted by Γ_n. If $G(\mathcal{T})$ is a complete graph, Cay(Sym(n), $\mathcal{T}$) is denoted by CT_n. In particular, if $G(\mathcal{T})$ is a star, Γ_n is the star graph S_n. If $G(\mathcal{T})$ is a path, Γ_n is the bubble sort graph B_n. The complete transposition graphs CT_3 and CT_4 are depicted in Fig. 1.

Complete-transposition graphs have been shown to have many desirable properties such as strong hierarchy, high connectivity, small diameter and average distance (see [10,14]), which makes it favorable as a network topology. It follows from the definition that CT_n is a $n(n-1)/2$-regular graph on $n!$ vertices. Lakshmivarahan et al. proved that CT_n is both vertex transitive and edge transitive [8]. It is well known that every edge transitive graph is maximally connected [15], that is, $\kappa(G) = \delta(G)$, where $\delta(G)$ is the minimum degree of graph G. Hence $\kappa(CT_n) = n(n-1)/2$. The girth of complete-transposition graphs is 4 (since modified bubble sort graphs is subgraph of complete-transposition graphs. The girth of modified bubble sort graph is 4 [18]).

In [3], Cheng et al. studied a class of Cayley graphs that are generated by transpositions. One of their results is as follows:

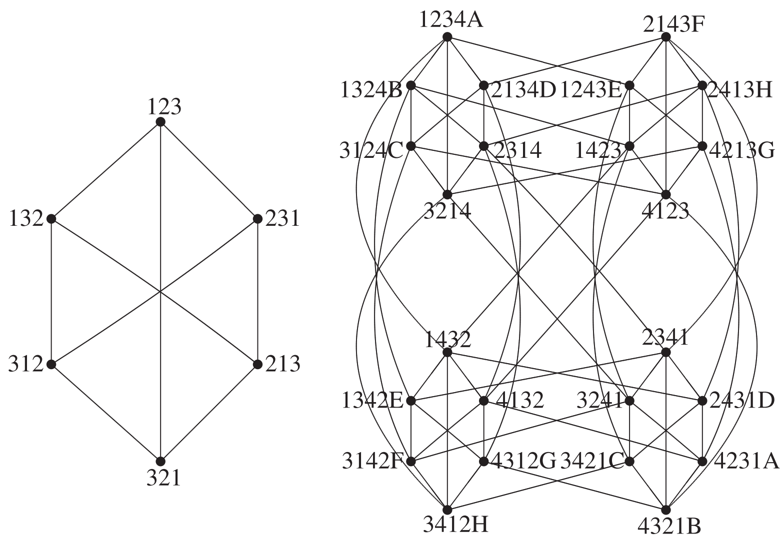

Fig. 1. Complete-transposition graphs CT_3 and CT_4.