Information Sciences 294 (2015) 127–138

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

The behavioral meaning of the median

Inés Couso [a,*], Serafín Moral [b], Luciano Sánchez [c]

[a] Dep. of Statistics and O.R., University of Oviedo, Spain
[b] Dep. of Computer Sciences and A.I., University of Granada, Spain
[c] Dep. of Computer Sciences and A.I., University of Oviedo, Spain

ARTICLE INFO

Article history:
Received 24 May 2013
Received in revised form 1 July 2014
Accepted 22 August 2014
Available online 22 September 2014

Keywords:
Stochastic ordering
Statistical preference
Imprecise Probability
Desirable gamble
Median

ABSTRACT

We extend the notion of statistical preference to the general framework of imprecise probabilities, by proposing a new notion of desirability of gambles called "sign-desirability", different from the usual desirability notion in Walley's framework. We axiomatically characterize coherent families of sign-desirable gambles. We furthermore prove that the pair of lower and upper previsions of a gamble, according to this new desirability notion, coincides with the pair of bounds (infimum and supremum) of the set of medians associated to a coherent family of linear previsions. Thus, a general notion of median is naturally derived, and provided with a behavioral meaning. As a consequence of these results, the connection between statistical preference in classical Probability Theory, and the sign of the median of the difference of two random variables is laid bare.

1. Introduction

In the field of decision making, several criteria of preference between random variables have been proposed in the literature, like for instance stochastic dominance [14] or dominance in the sense of expected utility [20]. One of them, called *statistical preference* [7,9] is based on Condorcet's voting criterion. According to Condorcet [5], an alternative that defeats every other in pairwise simple majority voting is the socially optimal choice. In accordance with this idea, in a numerical setting, if we consider a random vector (X, Y), and we identify the probability of occurrence of the event $X > Y$ with the proportion of voters supporting X against Y, the random variable X is said to be statistically preferred to Y when the probability that X takes a value greater than Y is not lower than the probability that Y exceeds X, i.e., when $P(X > Y) \geqslant P(Y > X)$. In this paper, we aim to extend the notion of statistical preference to the Theory of Imprecise Probabilities [22,15,2], relating it to the notions of *desirability* and *preference* between gambles. An initial attempt was made in [18,19], where statistical preference was transferred from Probability to Possibility Theory. When proposing the general formulation in Imprecise Probabilities, the problem of reconciling two different ways of treating preference relations will arise. In fact, a preference relation for pairs of random variables (or *gambles*) can be understood in two different ways:

- The expert's information is initially assessed by means of a (partial) preference relation between gambles. As a consequence, a set of joint feasible *linear previsions* is derived from it. This is the approach followed in the general Theory of Imprecise Probabilities (see [6,22,15]).

* Corresponding author. Tel.: +34 985181906; fax: +34 985181902.
E-mail addresses: couso@uniovi.es (I. Couso), smc@decsai.ugr.es (S. Moral), luciano@uniovi.es (L. Sánchez).

http://dx.doi.org/10.1016/j.ins.2014.08.049
0020-0255/

- The joint probability associated to every pair of gambles is assumed to be (partially) known, and a preference relation is derived from it. This is the approach considered in [9,11,12,18,19], for instance.

Taking into account the above duality, we will first start from a coherent set of desirable gambles [23,6] and we will say that a gamble X is sign-preferred to another gamble Y when the sign of their difference is a desirable gamble. Afterwards, we will show that sign-almost-preference becomes statistical preference, when the credal set induced by the coherent family of desirable gambles is a singleton. In other words, we will show that almost-sign-preference formally extends the notion of statistical preference. In a second step, we will consider sign-preference and sign-desirability as primary concepts, appealing to a new idea of desirability: X will be said to be sign-desirable when we have stronger beliefs about $X > 0$ than about $X < 0$. In words, we accept the gamble X, because we have stronger beliefs on making money than on loosing it –no matter how much money–. We will provide a collection of axioms characterizing "coherent" sets of sign-desirable gambles. Based on this new desirability notion, we can define the lower prevision of X as the supremum of the constants c satisfying that $X - c$ is sign-desirable. This supremum makes sense as a threshold for buying prices: for any strictly lower price, you have stronger beliefs on earning money that on loosing it. Analogously, we will define the upper prevision as the infimum selling price. Once both alternative approaches are considered (namely, the notion of sign-desirability viewed as a secondary concept derived from usual desirability in Walley's setting [22], or as a primary concept), we will relate both, and derive an interesting conclusion: the infimum and supremum of the set of medians associated to a credal set coincide with the pair of lower and upper previsions induced by the set of strictly-sign-desirable gambles induced by that credal set. Thus, we derive, in a natural way, the general notion of median, providing it with a meaningful behavioral interpretation. As a consequence, we will be able to show that there exists a very strong connection between the relation of statistical preference of two random variables and the sign of the median of their difference. This result adds another piece to the puzzle about the relationships between different stochastic preferences proposed in the literature.

Statistical preference can be seen as an alternative to the expected utility model when the rewards are expressed in a qualitative scale [13]. So, it can be applied in any decision problem in which we only know an order between the utility values. This is also the case of the median as an alternative to mathematical expectation. The median is used as a robust alternative to the expectation in regression [17] or to signal processing using median filters [3]. The results in this paper allow a principled extension of these concepts and associated procedures to the general case of Imprecise Probabilities.

The paper is organized as follows: Section 2 introduces the basic concepts of desirability and their relationships with credal sets and upper and lower previsions; Section 3 studies the concept of sign-desirability and its connections with statistical preference. It also proposes a set of axioms for coherent sign-desirable sets of gambles; Section 4 introduces the concept of median starting from coherent sets of desirable gambles and also contains a preliminary study of choice functions based on sign-desirability and the median; finally Section 5 is devoted to the conclusions.

2. Sets of desirable gambles and partial preferences

Let Ω denote the set of outcomes of an experiment. A *gamble*, X, on Ω is a bounded mapping from Ω to $\mathbb{R}$ (the real line). If you were to accept gamble X and ω turned to be true, then you would gain $X(\omega)$. (This reward can be negative, and then it will represent a loss.) Let $\mathcal{L}$ denote the set of all gambles (bounded mappings from Ω to $\mathbb{R}$). A subset $\mathcal{D}$ of $\mathcal{L}$ is said to be a *coherent set of desirable gambles* [22] when it satisfies the following four axioms:

D1. If $X \leqslant 0$, then $X \notin \mathcal{D}$ *(avoiding partial loss)*,
D2. If $X \in \mathcal{L}, X \geqslant 0$ and $X \neq 0$, then $X \in \mathcal{D}$ *(accepting partial gain)*,
D3. If $X \in \mathcal{D}$ and $c \in \mathbb{R}^+$, then $cX \in \mathcal{D}$ *(positive homogeneity)*,
D4. If $X \in \mathcal{D}$ and $Y \in \mathcal{D}$, then $X + Y \in \mathcal{D}$ *(Addition)*.

For a detailed justification of each of the above axioms concerning coherence in assessments of a subject, we refer the reader to (cf. [22], Section 2.2.4).

Example 1. Let us consider a frame with three elements $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and the following gambles defined on it:

$$X_1(\omega_1) = 2, X_1(\omega_2) = -1, X_1(\omega_3) = -1$$
$$X_2(\omega_1) = -1, X_2(\omega_2) = 2, X_2(\omega_3) = -1$$

Let us consider the following family of gambles containing X_1 and X_2:

$$\mathcal{D} = \{X : X \geqslant \alpha_1 X_1 + \alpha_2 X_2, \alpha_1 > 0, \alpha_2 \geqslant 0\} \cup \{X : X \geqslant 0, X \neq 0\}.$$

It is immediate to show that Axioms D1,D2,D3, and D4 are satisfied.

The *lower prevision induced by a set of desirable gambles* $\mathcal{D}$ is the set function $\underline{P} : \mathcal{L} \to \mathbb{R}$ defined as follows:

$$\underline{P}(X) = \sup\{c : X - c \in \mathcal{D}\}. \tag{1}$$