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Induced weighted operators based on dissimilarity functions ☆

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ABSTRACT

Based on the minimization considering dissimilarity function $D(x,y) = (f(x) - f(y))^2$ induced ordered weighted averaging operators *IOWA* and induced ordered generalized mixture operators $IOM_{\mathbf{g}}$ are discussed. In general, these operators need not be monotone non-decreasing, thus they need not be the aggregation operators. This paper introduces conditions for their monotonicity and weak monotonicity. Several examples are included.

1. Introduction and preliminaries

This paper is a natural continuation of our study of weighted operators based on dissimilarity function $D(x,y) = (f(x) - f(y))^2$ and their monotonicity in the paper Špirková [21]. The mentioned paper discusses arithmetic means, ordered weighted averaging *OWA* operators and mixture operators $M_{\mathbf{g}}$ as operators $A_{\mathbf{w},D},\ A_{\mathbf{g},D}$, where $\mathbf{w}$ is a weighting vector, $\mathbf{g}$ is a vector of weighting functions. Special attention is paid to the monotonicity of mixture operators. The mentioned operators were discussed, e.g. in Calvo and Beliakov [6], Calvo et al. [7], and Yager [24,26].

In this paper we introduce a generalization of induced ordered weighted averaging *IOWA* operators and induced ordered generalized mixture operators $IOM_{\mathbf{g}}$ on the basis of minimization with respect to dissimilarity function $D(x,y) = (f(x) - f(y))^2$. In general, these operators need not be monotone non-decreasing. Therefore, in this paper we introduce sufficient conditions for their monotone and weak monotone non-decreasingness with respect to Wilkin and Beliakov [23]. *IOWA* operators were introduced by Yager and Filev [25] and subsequently they were discussed and extended, for instance, in Merigó et al. [10,11], Špirková [16,17,19]. Induced ordered generalized mixture operators $IOM_{\mathbf{g}}$ were introduced in Špirková [19,20].

This paper is organized as follows. In Section 1, we briefly restate the basic definitions of aggregation functions and dissimilarity functions. In Section 2, we introduce a generalization of the *IOWA* operators and subsequently in SubSection 2.1. the conditions for their monotonicity and weak monotonicity. In Section 3, we present a generalization of the $IOM_{\mathbf{g}}$ operators. Moreover, in SubSection 3.1. we introduce the conditions for their monotonicity and weak monotonicity. Finally, in Section 4, we give conclusions.

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Now, we briefly restate basic considerations and definitions.

Let $I \subset \overline{R} = [-\infty, \infty]$ be a closed real non-trivial interval, $I = [a, b]$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an input vector. The interval I is usually some of the intervals $[0,1],\ [0,\infty]$. Following Beliakov et al. [1], Bustince et al. [2], Calvo et al. [8], Grabisch et al. [9] and Torra and Narukawa [22] we give the definition of aggregation function.

Definition 1.1. A function $A:\ [a,b]^n \to [a,b]$ is called an aggregation function if it is monotone non-decreasing in each variable and satisfies $A(\mathbf{a}) = a,\ A(\mathbf{b}) = b$, where $\mathbf{a} = (a, a, \ldots, a),\ \mathbf{b} = (b, b, \ldots, b)$.

We recall basic definitions of a monotonicity, weak monotonicity and shift-invariance with respect to Wilkin and Beliakov [23].

Definition 1.2. A function $A:\ [a,b]^n \to [-\infty, \infty]$ is monotone non-decreasing if and only if, $\forall\ \mathbf{x},\ \mathbf{y} \in [a,b]^n,\ \mathbf{x} \leqslant \mathbf{y}$ then $A(\mathbf{x}) \leqslant A(\mathbf{y})$.

Definition 1.3. A function $A:\ [a,b]^n \to [-\infty, \infty]$ is weakly monotone non-decreasing (or directionally monotone) if $A(\mathbf{x} + k\mathbf{1}) \geqslant A(\mathbf{x})\ \forall\ \mathbf{x}$ and for any $k > 0,\ \mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{\text{n times}})$, such that $\mathbf{x},\ \mathbf{x} + k\mathbf{1} \in [a,b]^n$.

Definition 1.4. A function $A:\ [a,b]^n \to [a,b]$ is shift-invariant (stable for translations) if $A(\mathbf{x} + k\mathbf{1}) = A(\mathbf{x}) + k\ \forall\ \mathbf{x}$ and for any $k > 0$, whenever $\mathbf{x},\ \mathbf{x} + k\mathbf{1} \in [a,b]^n$ and $A(\mathbf{x}) + k \in [a,b]$.

On the basis of Bustince et al. [2–4], Calvo and Beliakov [6], Mesiar et al. [12–14], we recall the definition of dissimilarity function on I.

Definition 1.5. Let $K : R \to R$ be a convex function with unique minimum $K(0) = 0$ and let $f : I \to R$ be a continuous strictly monotone function. Then the function $D : I^2 \to R$ given by $D(x,y) = K(f(x) - f(y))$ is called a dissimilarity function (on I).

On the basis of the properties of functions K and f a dissimilarity function satisfies the following axioms:

- $D(x,x) = 0$ (minimality),
- $D(x,y) = D(y,x) \geqslant 0$ (symmetry and nonnegativity),
- $D(x,y) \geqslant D(z,y)$ whenever $x \geqslant z \geqslant y$ or $x \leqslant z \leqslant y$ (monotonicity).

Typical examples of dissimilarity functions can be found in Calvo et al. [8], Mesiar et al. [13,14], and Špirková [21].

In this paper, we focus on the use of a dissimilarity function $D(x,y) = (f(x) - f(y))^2$. We deal with induced *OWA* operators and their extensions $I^{2n} \to I$ defined for each $n \in N$, an input vector of the argument variables $\mathbf{x} = (x_1, \ldots, x_n)$, an input vector of the order inducing variables $\mathbf{u} = (u_1, \ldots, u_n)$, functions $h : I \to I,\ H : I \to I$, weighting vector $\mathbf{w} = (w_1, \ldots, w_n),\ \mathbf{w} \neq \mathbf{0}$ and $w_i \geqslant 0,\ i = 1, \ldots, n$; continuous strictly monotone function $f : I \to [-\infty, \infty]$, continuous weighting function $g : I \to [0, \infty[$, and weighting vector of continuous weighting functions $\mathbf{g} = (g_1, \ldots, g_n),\ g_i : I \to [0, \infty[,\ i = 1, \ldots, n$.

2. Induced *OWA* operators

The induced ordered weighted averaging *IOWA* operators were introduced by Yager and Filev [25]. The *IOWA* operators represent an extension to the *OWA* operators. The reordering step is induced by another mechanism represented by a vector of order inducing variables $\mathbf{u}$, where the ordered position of the arguments x_i depends on the values of the order inducing variables u_i.

Firstly, we restate the basic definitions of *IOWA* and *Quasi-IOWA* operators. The *IOWA* operator can be defined as follows.

Definition 2.1. The operator *IOWA*: $[a,b]^n \to [a,b]$ given by

$$IOWA(\langle u_1, x_1\rangle, \ldots, \langle u_n, x_n\rangle) = \frac{\sum_{i=1}^n w_i x_{(i)}}{\sum_{i=1}^n w_i}, \tag{1}$$

where $x_{(i)} = x_j$ is the aggregated value from the pair $\langle u_j, x_j\rangle$, where the order inducing variable u_j is the i-th lowest one, i.e., $u_{(1)} < u_{(2)} < \cdots < u_{(n)}$, is called an induced ordered weighted averaging operator.

If there are some ties among the order inducing variables, we will consider all permutations σ satisfying $u_{\sigma(1)} \leqslant u_{\sigma(2)} \leqslant \cdots \leqslant u_{\sigma(n)}$ and we will them consider the arithmetic mean of all corresponding input values given by formula (1). This feature of induced operators will be assumed in all subsequent definitions.

The *Quasi-IOWA* operator was discussed in Merigó and Gil-Lafuente [11] and Špirková [16]. It can be defined as follows.

Definition 2.2. The operator *Quasi-IOWA* : $[a,b]^n \to [a,b]$ given by

$$Quasi\text{-}IOWA(\langle u_1, x_1\rangle, \ldots, \langle u_n, x_n\rangle) = f^{-1}\left(\frac{\sum_{i=1}^n w_i f(x_{(i)})}{\sum_{i=1}^n w_i}\right), \tag{2}$$