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Linear optimization problem constrained by fuzzy max–min relation equations

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ABSTRACT

Fang and Li introduced the optimization model with a linear objective function and constrained by fuzzy max–min relation equations. They converted this problem into a 0–1 integer programming problem and solved it using the jump-tracking branch-and-bound method. Subsequently, Wu et al. improved this method by providing an upper bound on the optimal objective value and presented three rules for simplifying the computation of an optimal solution. This work presents new theoretical results concerning this optimization problem. They include an improved upper bound on the optimal objective value, improved rules for simplifying the problem and a rule for reducing the solution tree. Accordingly, an accelerated approach for finding the optimal objective value is presented, and represents an improvement on earlier approaches. Its potential applications are discussed.

1. Introduction

This work considers the following linear objective function with a fuzzy relation equation.

$$\text{Min} \quad Z(x) = \sum_{i=1}^{m} c_i x_i, \tag{1}$$

$$\text{subject to } x \circ A = b, \tag{2}$$

where $x \in [0,1]^m$, $c_i \in R$ $A = [a_{ij}]_{m\times n}$ with $0 \leqslant a_{ij} \leqslant 1$, $b = (b_1, \ldots, b_n)$ is an n-dimensional vector with $0 \leqslant b_j \leqslant 1$, and "$\circ$" represents the max–min composition. Let $X(A,b) = \{x \in [0,1]^m | x \circ A = b\}$ be the solution set of (2). It is well-known that when $X(A,b) \neq \phi$, $X(A,b)$ can be completely determined by the unique maximum solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$, and a finite number of minimal solutions [2]. Here $\bar{x}_i = 1$ if $a_{ij} \leqslant b_j$ for all $1 \leqslant j \leqslant n$, otherwise $\bar{x}_i = \min\{b_j | a_{ij} > b_j\}$ [9]. We define $x^1 = (x_i^1)_{1\times m} \leqslant x^2 = (x_i^2)_{1\times m}$ if and only if $x_i^1 \leqslant x_i^2$ for all $1 \leqslant i \leqslant m$. A solution $\underline{x} \in X(A,b)$ is a minimal solution if any $x \in X(A,b)$, $x \leqslant \underline{x}$ implies that $x = \underline{x}$. A solution $x^* \in X(A,b)$ is optimal for problem (1) and (2) if $Z(x^*) \leqslant Z(x)$ for all $x \in X(A,b)$. For any solution $x = (x_i)_{1\times m} \in X(A,b)$, x_i is a binding variable if $\min(x_i, a_{ij}) = b_j$ holds for some $1 \leqslant j \leqslant n$.

To simplify the problem that is composed of Eqs. (1) and (2), as many decision variables x_i as possible should be optimized. Based on this fact, Fang and Li [3] separated problem (1) and (2) into two sub-problems:

$$\min \quad \sum_{i=1}^{m} c_i^1 x_i,$$
$$\text{subject to } x \circ A = b \tag{3}$$

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and

$$\begin{aligned} &\min \quad \sum_{i=1}^{m} c_i^2 x_i, \\ &\text{subject to } x \circ A = b, \end{aligned} \tag{4}$$

where

$$c_i^1 = \begin{cases} c_i & \text{if} \quad c_i < 0 \\ 0 & \text{if} \quad c_i \geqslant 0 \end{cases} \quad \text{and} \quad c_i^2 = \begin{cases} 0 & \text{if} \quad c_i < 0 \\ c_i & \text{if} \quad c_i \geqslant 0 \end{cases} \quad \text{for all} \quad 1 \leqslant i \leqslant m. \tag{5}$$

Clearly, the maximum solution $\bar{x}$ to (2) when it exists, solves problem (3), and one of the minimum solutions $\underline{x}^* = (\underline{x}_i^*)_{1\times m}$ to (2) solves problem (4) [4,8,12]. Therefore, the solution $x^* = (x_i^*)_{1\times m}$ defined by

$$x_i^* = \begin{cases} \underline{x}_i^* & \text{if} \quad c_i \geqslant 0 \\ \bar{x}_i & \text{if} \quad c_i < 0 \end{cases} \quad \text{for} \quad 1 \leqslant i \leqslant m,$$

solves problem (1) and (2) [3]. The maximum solution to (2) is easily determined, but finding the minimal solutions may be difficult. Algorithms exist for finding the set of minimal solutions to (2) [1,7,9–11,15]. Since the optimal solution to problem (4) is among the minimal solutions to (2), one possible way to find the optimal objective value is first to compute all minimal solutions to (2) and then, by enumeration, to find the optimal objective value. The other approach for finding the optimal objective value is the branch-and-bound method. Fang and Li [3] converted problem (4) into a 0–1 integer programming problem and solved it by the branch-and-bound method. Subsequently, Wu et al. [14] improved their method by providing a fixed upper bound on the optimal objective value. Wu and Guu [13] recently improved on that work by updating the "fixed" upper bound on the optimal objective value. Two numerical examples indicate that this improvement actually reduces the computational burden of finding the optimal objective value. This work improves upon the work of Wu and Guu [13]. More reducing rules are devised, and an improved initial upper bound is obtained. This improvement enables the problem to be solved more quickly than in previous work [3,13,14], in the sense that fewer nodes in the solution tree are visited.

The rest of the paper is organized as follows. Section 2 contains the primary results. Section 3 presents three examples that demonstrate the proposed method outperforms available approaches [3,13,14] and its application to the optimal three-tier multimedia streaming services problem. Section 4 finally draws conclusions.

2. Simplifying and solving the problem

This section firstly presents some rules for reducing problem (1) and (2), and then presents a procedure for solving it. For convenience, the coefficients c_i are arranged in increasing order and b_j arranged in decreasing order, such that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$; also $X(A,b) \neq \phi$ is assumed. The method of Fang and Li [3] is improved by setting $c_i^1 = c_i$ for $c_i \leqslant 0$ and $c_i^2 = c_i$ for $c_i > 0$ in (5). In this manner, $\bar{x}$ is found to solve problem (3), and one of the minimum solutions to (2) also solves problem (4). Since problem (3) is solved by $\bar{x}, x_i^* = \bar{x}_i$ for $c_i \leqslant 0$ is determined for any optimal solution $x^* = (x_i^*)_{1\times m}$ to problem (1) and (2). The $\bar{x}_i$ terms are utilized to reduce problem (4) as follows. Let $J_i = \{1 \leqslant j \leqslant n \mid \min(\bar{x}_i, a_{ij}) = b_j\},\ 1 \leqslant i \leqslant m$. For each $\bar{x}_i$, since $\min(\bar{x}_i, a_{ij}) = b_j, \bar{x}_i$ satisfies the jth equation for all $j \in J_i$. Thus, the size of problem (2) can be reduced by deleting row i, and column j of A and b_j, for all $j \in J_i$. Therefore, the following rule is obtained.

Rule 1. Delete row i, and column j of A and b_j, for all $j \in J_i$, and set $x_i^* = \bar{x}_i$ for any optimal solution $x^* = (x_i^*)_{1\times m}$, where $c_i \leqslant 0$.

Remark 1. Rule 1 is similar to case 1 in the work of Loetamonphong and Fang [6, p. 512] and Rule 1 in the work of Shieh [12], who considered this optimization problem with the max-product and the max-Archimedean t-norm compositions, respectively.

For convenience, the reduced constraint equation is also denoted by (2). The remaining task, which is to solve the problem that is composed of Eqs. (1) and (2) is to solve problem (4), constrained by the reduced equations (after Rule 1 has been applied). Let $X_0(A,b)$ be the set of minimal solutions to (2). Since only the optimal solution to problem (4) can be found in $X_0(A,b)$, two rules for simplifying the constraint equation are presented below. Let $I_j = \{i \in I \mid \min(\bar{x}_i, a_{ij}) = b_j\},\ j \in J$, where J is the column index set of the reduced matrix A.

Rule 2 [11]. If $I_k \subseteq I_l$ and $b_k \geqslant b_l$, then deleting column l of A and b_l does not affect $X_0(A,b)$.

Remark 2. Wu and Guu [13] restricted Rule 2 in the case of $I_k = \{i\}$, a singleton set. Clearly, Rule 2 is an extension of the corresponding rule of Wu and Guu [13].

Lemma 1 [11]. *If* $\underline{x} = (\underline{x}_i)_{i\in I} \in X_0(A,b)$ *and* $\min(\underline{x}_i, a_{ij_i}) = b_{j_i}$, *then* $i \in I_{j_i}$ *and min* $(\underline{x}_i, a_{ik}) = b_k$, *for all* $k > j_i$ *and* $i \in I_k$, *where* $b_{j_i} = \max\{b_j \mid \min(\underline{x}_i, a_{ij}) = b_j,\ j \in J\}$.