Information Sciences 234 (2013) 86–96

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Resolution of a system of the max-product fuzzy relation equations using $L\circ U$-factorization

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ARTICLE INFO

Article history:
Available online 13 April 2011

Keywords:
Fuzzy relation equation
Max-product operator
$L\circ U$-factorization

ABSTRACT

In this paper, the LU-factorization is extended to the fuzzy square matrix with respect to the max-product composition operator called $L\circ U$-factorization. Equivalently, we will find two fuzzy (lower and upper) triangular matrices L and U for a fuzzy square matrix A such that $A = L\circ U$, where "$\circ$" is the max-product composition. An algorithm is presented to find the matrices L and U. Furthermore, some necessary and sufficient conditions are proposed for the existence and uniqueness of the $L\circ U$-factorization for a given fuzzy square matrix A. An algorithm is also proposed to find the solution set of a square system of Fuzzy Relation Equations (FRE) using the $L\circ U$-factorization. The algorithm finds the solution set without finding its minimal solutions and maximum solution. It is shown that the two algorithms have a polynomial-time complexity as $O(n^3)$. Since the determination of the minimal solutions is an NP-hard problem, the algorithm can be very important from the practical point of view.

1. Introduction

The notion of FRE is associated with the composition of fuzzy binary relations. The FRE have been intensively investigated both from a theoretical standpoint and in view of applications since they were first introduced by Sanchez [53,54]. The FRE play important roles in many applications, such as intelligence technology [11,17], image reconstruction [16,31,39,40], etc. Therefore, how to compute the solutions of FRE is a fundamental problem.

The solvability criteria of max-T equations were first established by Sanchez [54] for the max–min equations and then extended by Pedrycz [42,43] and Miyakoshi and Shimbo [37]. Recently, there have been many research papers investigating the solvability of the FRE, by generalizing and extending the original results of [52,54] in various directions [1–10,12–70]. The structure of the complete solution set of the max–min equations was first characterized by Sanchez [55] and generalized to the max-T equations by Di Nola et al. [14,15]. It is well-known that the complete solution set of a consistent finite system of sup-T equations can be determined by a maximum solution and a finite number of minimal solutions. The consistency of a system of the max-T equations can be easily verified by checking the potential maximum solution.

Higashi and Klir [22] derived several alternative general schemes to find the solutions. Moreover, the resolution problem of a finite system of the max-T equations with a general continuous triangular t-norm is discussed by Li and Fang [27]. Various methods have been developed to detect the minimal solutions for the max-T equations with a specific triangular norm. Approaches based on some type of the quasi- characteristic matrix were proposed by Peeva [44,45], Han and Sekiguchi [21], Li [25], Wang and Hsu [62] and Peeva and Kyosev [46,48]. Some rule-based methods were proposed by Arnould and Tano [3,4]. In 1988, Lichung and Boxing [28] introduced an algebraic method to calculate all minimal solutions. Later, De Baets [13] provided an analytical method to find them. In 2002, Louh et al. [32] used the matrix

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doi:10.1016/j.ins.2011.04.012

pattern to compute graphically the minimal solutions. Peeva [47] proposed a universal algorithm which improves the algebraic method. Yeh [70] used the covering matrix, which was first introduced by Lichung and Boxing [28], to develop a new algorithm for the computation of all minimal solutions of the max–min FRE. Various estimates of the number of the minimal solutions can be found in [12,45–48,56,59,63]. More discussions on the minimal solutions may be found in [6,7,12,14,22,23,30,32,33, 38,41,50,65,68,69]. However, as shown in [9,10,29,35,36], the detection of all the minimal solutions is closely related to the *set covering problem* and hence, it is an NP-hard problem. Therefore, designing an algorithm for the resolution of a system of FRE without finding all its minimal solutions is motivated. We now focus on the resolution of the square system of FRE. It can be interesting and useful to extend the method to a more general class of FRE.

The LU-factorization method in the linear algebra with the summation and product operations is very interesting and useful for the reduction of a non-sparse square matrix to the product of two sparse square matrices called lower and upper triangular matrices. The method is applied to solve a non-sparse square linear system. Using the LU-factorization method, the non-sparse square linear system is easily reduced to two sparse square linear systems as lower and upper triangular systems. Hence, we can easily solve the systems by the forward and backward substitutions. This procedure is called the resolution of the system by the LU-factorization. Thus the study of the topic for a square system of the max-product FRE is motivated. We are now interested to answer the following questions about the non-sparse square system of FRE.

(1) Under what conditions is there some such factorization for a fuzzy square matrix with respect to the max-product operator?
(2) If there exists such a factorization, then when is it unique?
(3) Can the factorization method be extended to a fuzzy square matrix with respect to the max-product operator?
(4) Can the procedure be extended to solve a system of FRE with respect to the max-product composition operator?
(5) Are there efficient algorithms with polynomial-time complexities to do the mentioned works in the questions (3) and (4)?

Therefore, the study of the LU-factorization for the systems of FRE with the max-product operator is motivated. Furthermore, answering to the above questions is another motivation for this study.

In this paper, we first extend the LU-factorization to a fuzzy square matrix with respect to the max-product composition operator. In other words, we will find two fuzzy (lower and upper) triangular matrices L and U for a fuzzy square matrix A such that $A = L \circ U$, where "$\circ$" is the max-product composition. The LU-factorization for a fuzzy square matrix with respect to the max-product composition is called an $L\circ U$-factorization. Then, an algorithm is proposed to find an $L\circ U$-factorization of the fuzzy matrix A using the special structure of the matrices L and U and the max-product composition operator. Its time computational complexity is then computed. It is also shown that the algorithm has a polynomial-time complexity as $O(n^3)$. Furthermore, some necessary and sufficient conditions are proposed for the existence and uniqueness of the $L\circ U$-factorization for a given fuzzy matrix A with respect to the max-product composition. The $L\circ U$-factorization is applied to find the solution set of a square system of FRE. In this process, using the $L\circ U$-factorization structure, the square system is easily reduced to two sparse square systems as lower and upper triangular systems. Hence, we can easily solve the systems by the forward and backward substitutions. By combining the above algorithm, the forward and backward substitutions, another algorithm is proposed to find the solution set of the system of FRE. It is shown that the algorithm has the same polynomial-time complexity as the first algorithm, i.e., $O(n^3)$. With respect to the other methods, we are able to find the solution set of FRE without finding its minimal solutions.

This paper is organized as follows. Section 2 studies the $L\circ U$-factorization of a fuzzy matrix with respect to the max-product composition. An algorithm is also presented to find the matrices L and U. Furthermore, its time computational complexity is computed. Section 3 presents the necessary and sufficient conditions for the existence and uniqueness of the $L\circ U$-factorization for a given fuzzy square matrix. In Section 4, an algorithm is proposed to solve the system of the max-product FRE using the $L\circ U$-factorization. Its computational complexity is also computed. The algorithm is outlined and illustrated by an example. The conclusions are finally given in Section 5.

2. *L*∘*U*-factorization of a fuzzy matrix with respect to the max-product composition operator

Let $A = [a_{ij}]$, $0 \leqslant a_{ij} \leqslant 1$, be an $n \times n$-dimensional fuzzy matrix. We will show how to decompose a fuzzy square matrix A into two fuzzy (lower and upper) triangular matrices L and U such that $A = L\circ U$, where the symbol "$\circ$" denotes the max-product composition operator. In other words, we try to find two fuzzy (lower and upper) triangular matrices $L = [l_{ij}]_{n\times n}$ and $U = [u_{ij}]_{n\times n}$ such that $0 \leqslant l_{ij}, u_{ij} \leqslant 1$,

$$l_{ij} = \begin{cases} l_{ij}, & i > j, \\ 1, & i = j, \quad \forall\, i,j \in \underline{n}, \\ 0, & i < j, \end{cases} \tag{1}$$