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Real coextensions as a tool for constructing triangular norms

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ABSTRACT

We present in this paper a universal method of constructing left-continuous triangular norms (l.-c. t-norms). The starting point is an arbitrary, possibly finite, totally ordered monoid fulfilling the conditions that are characteristic for l.-c. t-norms: commutativity, negativity, and quanticity. We show that, under suitable conditions, we can extend this structure by substituting each element for a real interval. The process can be iterated and if the final structure obtained in this way is order-isomorphic to a closed real interval, its monoidal operation can, up to isomorphism, be identified with a l.-c. t-norm.

We specify the constituents needed for the construction in an explicit way. We furthermore illustrate the method on the basis of a number of examples.

1. Introduction

A fundamental issue in fuzzy set theory has been the question how the basic set-theoretical operations of intersection, union, and complement should be generalised to the case that membership comes in degrees; see, e.g., [15, Chapter 10]. By general agreement, such operations should be defined pointwise. Indeed, this basic assumption is in accordance with the disjunctive interpretation of fuzzy sets and arguments on axiomatic grounds have been given, e.g., in [16]. Otherwise, however, no standards exist and there are not even convincing arguments for restricting the possibilities to a manageable number in order to facilitate the choice.

To define the intersection of fuzzy sets, we are hence in need of a suitable binary operation on the real unit interval [0, 1]. It is clear that only triangular norms (t-norms) come into question. The minimal requirements of a conjunction are then fulfilled: associativity, commutativity, neutrality w.r.t. 1, and monotonicity in each argument. However, these properties are not very specific and t-norms exist in abundance. In fuzzy set theory, t-norms have consequently become a research field in its own right. A basic reference is the monograph [17] and overviews are provided, e.g., in [6,20]. Among the numerous specialized studies on t-norms from recent times, we may mention, e.g., [7,22]. The present paper is meant as a further contribution towards a better understanding of these operations.

Also in mathematical fuzzy logic, t-norms are employed for the interpretation of the conjunction. In this context, the implication is commonly assumed to be the adjoint of the conjunction; see, e.g., [11]. Given a t-norm, however, a residual implication does not necessarily exist; to this end, the t-norm must be in each argument left-continuous. Here, we generally assume this additional property to hold.

To explore t-norms, different perspectives can be chosen. Often t-norms have been studied as two-place real functions and geometric aspects have played a major role. For the sake of a classification, it makes sense not to distinguish between isomorphic operations. We recall that t-norms $\odot_1$ and $\odot_2$ are isomorphic if there is an order automorphism ϕ of the real

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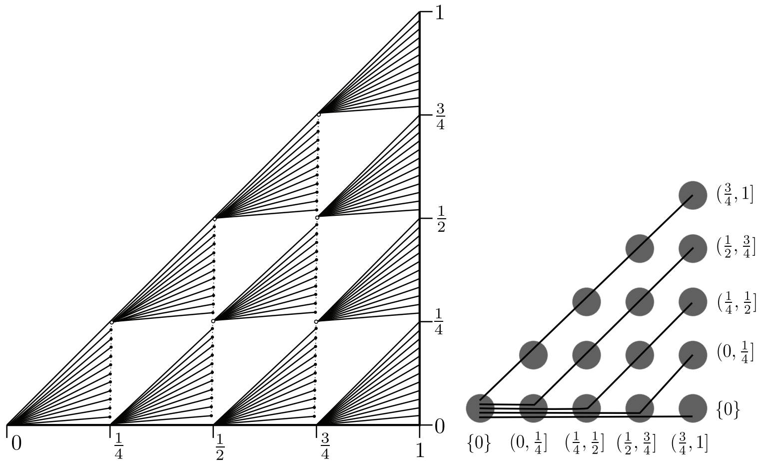

Fig. 1. Left: The t-norm $\odot_H$. We show a selection of vertical cuts, that is, the multiplication with certain fixed elements. Right: The quotient w.r.t. $(\frac{3}{4}, 1]$ is the five-element Łukasiewicz chain.

unit interval such that $a \odot_2 b = \phi^{-1}(\phi(a) \odot_1 \phi(b))$ for any $a, b \in [0, 1]$. In this case, it is reasonable to adopt an algebraic perspective. We will do so as well. Our aim is to classify, up to isomorphism, the structures $([0, 1]; \leq, \odot, 1)$, where $\leq$ is the natural order of the reals and $\odot$ is a l.-c. t-norm.

The structures of the form $([0, 1]; \leq, \odot, 1)$ are totally ordered monoids, or tomonoids for short [5,8]. They are, however, quite special among this type of algebras. The tomonoids in which we are interested are commutative, because so is each t-norm. Furthermore, they are negative, because the monoidal identity is the top element. Finally, the left-continuity of t-norms corresponds to a property that we call “quantic”. In a word, the quantic, negative, and commutative tomonoids (q.n.c. tomonoids) whose base set is the real unit interval, are in a one-to-one correspondence with l.-c. t-norms.

Investigating tomonoids, we can profit from the enormous progress that the research on algebraic structures around left-continuous t-norms has made in recent times. A summary of results on residuated structures can be found, e.g., in [9, Chapter 3]. In [24], t-norms are especially taken into account. A number of further, more specialised overviews is contained in [4]. We may in fact say that the situation as regards t-norms is today considerably more transparent than it used to be a few years ago.

To reveal the structure of a t-norm $\odot$, the first natural step is to determine the quotients of the tomonoid based on $\odot$. In fact, it has turned out that in this way the vast majority of t-norms known in the literature can be described in a uniform and transparent way. A systematic review of t-norms from this perspective was undertaken in [27] and in [28] the approach was applied in order to systematize a number of well-known t-norm construction methods.

Let us provide, on an intuitive basis, a summary of what follows. The quotients of tomonoids in which we are interested are constructed as follows. Let $\odot$ be a l.-c. t-norm and assume that the set $F \subseteq [0, 1]$ is (i) of the form $(d, 1]$ or $[d, 1]$ for a $d \in [0, 1]$, and (ii) closed under multiplication, that is, $a \odot b \in F$ for any $a, b \in F$. Then we call F a filter. Define the equivalence relation $\sim_F$ by requiring $a \sim_F b$ if $a \odot f \leq b$ and $b \odot f \leq a$ for some $f \in F$. Then $\sim_F$ is a congruence of the tomonoid $([0, 1]; \leq, \odot, 1)$. This means that $[0, 1]$ is partitioned into subintervals and $\odot$ induces a binary operation making the set of these subintervals into a tomonoid again.

Consider, e.g., the t-norm $\odot_H$ shown in Fig. 1 (left), which is a modification of a t-norm defined by Hájek [12]. We depict $\odot_H$ by indicating the mappings $[0, 1] \to [0, 1], \quad x \mapsto a \odot_H x$ for several $a \in [0, 1]$. We observe that $(\frac{3}{4}, 1]$ is closed under the operation $\odot$ and hence a filter. Forming the quotient by $(\frac{3}{4}, 1]$ leads to the partition $\{0\}$, $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, $(\frac{3}{4}, 1]$ of $[0, 1]$. The operation $\odot$ endows these five elements with the structure of the five-element Łukasiewicz chain L_5; see Fig. 1 (right).

We hence see that a seemingly complex t-norm possesses a quotient that is as simple as the five-element Łukasiewicz chain.

The question that we raise in the present work concerns the converse procedure. Let the tomonoid L_5 be given and assume that we want to expand each non-zero element to a left-open right-closed interval. Can we determine the monoidal operations on this enlarged universe such that the quotient is L_5? Are there any other operations apart from $\odot_H$ with this property? Under suitable conditions, we will present in this paper a way of determining all possible monoidal operations on the enlarged universe. The t-norm $\odot_H$ will turn out not to be the only solution and we easily determine the remaining ones as well.

Algebraically, our problem reads as follows. Let $(\mathcal{P}; \leq, \odot, 1)$ be a q.n.c. tomonoid. Extend the chain $\mathcal{P}$ by substituting each element for a left-open or left-closed, right-open or right-closed real interval. Let $\mathcal{L}$ be the enlarged chain and let F be the subset of $\mathcal{L}$ that has been chosen to replace the top element of $\mathcal{P}$. Our aim is to make $\mathcal{L}$ into a q.n.c. tomonoid of which F is a filter and whose quotient by F is the original tomonoid $\mathcal{P}$. To this end, we consider two situations. First, we assume