Information Sciences 370–371 (2016) 256–269

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

On the measure based formulation of multi-criteria decision functions

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A R T I C L E I N F O

Article history:
Received 4 September 2015
Revised 4 May 2016
Accepted 19 July 2016
Available online 25 July 2016

Keywords:
Decision-making
Choquet integral
Prioritized aggregation
Fuzzy measure
Multi-criteria

A B S T R A C T

The basic features and properties of a fuzzy measure are first introduced. We next discuss the Choquet integral and describe its usage in finding the aggregation of a collection of values guided by a fuzzy measure. We then show how the Choquet integral and an associated fuzzy measure can be applied to the construction of multi-criteria decision functions and describe how the associated fuzzy measure is used to model the desired multi-criteria decision function. We then look at various fuzzy measures and investigate the types of decision functions they allow us to formulate. Notable among the measures investigated here are those that allow us to model a prioritized aggregation of the criteria and a rule based aggregation of multiple criteria. Finally we show how to model linguistically specified decision functions using fuzzy measures.

1. Introduction

Multi-criteria decision-making is a very common task that is pervasive in our personal as well as professional life. Tasks as diverse as deciding what movie to see, allocating billions of dollars in project investments and deciding whether an incoming object is an enemy missile all have aspects of multi-criteria decision-making. Fuzzy set technology starting with the pioneering work by Bellman and Zadeh [1] has a long history of interest in this problem. One approach to multi-criteria decision-making is to construct a decision function by aggregating an alternative's satisfaction to the individual criteria and then selecting the alternative with the largest aggregated value [1]. Our focus here is on the formulation of multi-criteria decision functions based on the use of a fuzzy measure (monotonic set measure) [2–5]. Once having this measure we use the Choquet integral [6–9] to evaluate the satisfaction of the decision function for a given alternative solution. This approach essentially provides an average of the individual criteria's satisfaction to a given alternative, where the type of average is determined by the fuzzy measure used. In this work we show how the fuzzy measure can be used to model various types of complex relationships between the criteria. We see here that the fuzzy measure is an extremely useful tool in the domain of computational intelligence as it enables the transference of human cognitive ideas about the relationships between multiple criteria in a decision task into a formal mathematical model that can be used to compare various available alternatives

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http://dx.doi.org/10.1016/j.ins.2016.07.045

2. Fuzzy measures

A monotonic or fuzzy measure [10,11] on a space $X=\{x_1,\ldots, x_n\}$ is a set mapping $\mu: 2^X \to [0, 1]$ such that: **1)** $\mu(\emptyset)=0$, **2)** $\mu(X)=1$ and **3)** $\mu(A) \geq (B)$ if $A \supseteq B$. A fuzzy measure associates with subsets of X a value in the unit interval. In the following we simply refer to this as a measure.

If μ_1 and μ_2 are two measures such that $\mu_1(A) \geq \mu_2(A)$ for all A we denote this as $\mu_1 \geq \mu_2$. Two notable measures are μ^* and μ_*, defined respectively as $\mu^*(\emptyset)=0$ and $\mu^*(A)=1$ for all $A \neq \emptyset$ and $\mu_*(X)=1$ and $\mu_*(A)=0$ for all $A \neq X$. These are bounding measures, for any measure μ we have $\mu_* \leq \mu \leq \mu^*$.

We say a measure μ is a binary measure if $\mu(A) \in \{0, 1\}$ for all A.

If μ is a measure then a set mapping $\hat{\mu}$ defined so that $\hat{\mu}(A)=1-\mu(\bar{A})$ is called the dual of A. If is easy to see that $\hat{\mu}$ is itself a measure. Furthermore, since $\hat{\hat{\mu}}=\mu$ thus duals come in unique pairs. It can be shown that μ^* and μ_* are duals. We note that if μ is a measure then the set mapping μ_1 defined so that $\mu_1(A)=1-\mu(A)$ is not a measure. This is clear since $\mu_1(X)=1-\mu(X)=0$ and $\mu_1(\emptyset)=1-\mu(\emptyset)=1$.

In [12] we provided a correspondence between a measure μ and a fuzzy subset F_μ on the space 2^X. In particular, for any subset A of X if $F_\mu(A)=\mu(A)$ the F_μ is then called the fuzzy subset induced by the measure μ.

In [13] we discussed methods for obtaining a measure from other measures. A very important method involves the use of an aggregation function.

Definition. A mapping Agg: $I^q \to I$ is called an aggregation function [2,14] if:

1. $Agg(1, .., 1)=1$, **2.** $Agg(0, .., 0)=0$ and **3.** $Agg(a_1, .., a_q) \geq Agg(b_1,\ldots, b_q)$ if $a_i \geq b_i$ for all i.

An aggregation function is called a conjunctive aggregation function if $Agg(a_i,\ldots, a_q) \leq Min_j[a_j]$.

An example of a conjunctive aggregation function is the Min, $Agg(a_1,\ldots, a_q)=Min_j[a_j]$. Semantically the conjunctive aggregation operator is providing an ***anding*** of its arguments.

An aggregator is called a disjunctive aggregation operator if $Agg(a_1,\ldots, a_q) \geq Max_j[a_j]$. A prototypical example of a disjunctive aggregation operator is the Max, $Agg(a_1,\ldots, a_q)=Max_j[a_j]$. Conceptually a conjunctive aggregation operator is providing an ***oring*** of its arguments.

An aggregation function is called a mean if $Min_j[a_j] \leq Agg(a_1,\ldots, a_q) \leq Max_j[a_j]$. An prototypical example of the mean operator is the simple average $Agg\ (a_1,\ldots,a_q)=\frac{1}{q}\sum_{j=1}^{q} a_j$.

Let μ_j for j=1 to q be a collection of measures on X. Assume μ is a set mapping on X defined so that for any subset A of X, $\mu(A)=Agg(\mu_1(A), \mu_2(A),\ldots, \mu_q(A))$, it can be shown that μ is a measure on X [13,15]. That is, $\mu(\emptyset)=0$, $\mu(X)=1$ and μ is monotonic.

Thus here we see how we can obtain a new measure by aggregating other measures. For simplicity in the following we shall represent this operation as $\mu=Agg(\mu_1,\ldots, \mu_q)$.

Another approach for obtaining a new measure from another measure involves the use of a function f: $[0, 1] \to [0, 1]$ having the properties: **1)** $f(0)=0$, **2)** $f(1)=1$ and **3)** $f(a) \geq f(b)$ if $a \geq b$. Assume μ is a measure on X and let μ_1 be a set mapping defined so that $\mu_1(A)=f(\mu(A))$ for all subsets A of X then μ_1 is itself a measure. We shall refer to this operation as $\mu_1=f(\mu)$.

Some examples of f that satisfy the necessary conditions are: **(1)** $f(y)=y^\alpha$ for $\alpha>0$, **(2)** $f(y)=Min[ay, 1]$ for $a \geq 1$, **(3)** $f(y)=\sin(\frac{\Pi}{2}y)$, **(4)** f_α defined so that $f_\alpha(y)=0$ if $y \leq \alpha$ and $f_\alpha(y)=1$ if $y>\alpha$, **(5)** another function is a quasi-linear function: $f(y)=0$ for $y \leq \alpha$; $f(y)=\frac{y-\alpha}{\beta-\alpha}$ for $\alpha<y<\beta$; $f(y)=1$ for $y \geq \beta$

3. Choquet integral

One use of fuzzy measures is in the Choquet integral [2,16] which provides a way of aggregating a collection of values. In this application the measure determines the type of aggregation being performed.

Assume $X=\{x_1,\ldots, x_n\}$ are a collection of objects and μ is a measure on X. In addition associated with each x_i is a value $V_i \in R$. The Choquet integral of the collection $(V_1,\ldots, V_n)$ guided by the measure μ is defined as $Choq_\mu(V_1,\ldots, V_n)=\sum_{j=1}^{n}(\mu(H_j)-(\mu(H_{j-1}))\ V_{id(j)}$ where id is an index function so that $V_{id(j)}$ is the *j*th largest of the V_i, $V_{id(j)} \geq V_{id(j+1)}$ and $H_j=\{x_{id(k)} / k=1 \text{ to } j\}$, it is the subset of objects in X with the j largest values for V_i.

It can be show that we can express the Choquet integral as $Choq_\mu(V_1,\ldots, V_n)=\sum_{j=1}^{n}(\mu(H_j)(V_{id(j)}-V_{id(j+1)})$ with the convention that $V_{id(n+1)}=0$.

We note that the H_j form a chain on X, that is, $\emptyset=H_0 \subset H_1 \subset H_2 \ldots \subset H_n=X$. In addition we have $Card(H_j)=j$.

Essentially the Choquet integral provides a kind weighted average of its arguments, the V_j.We also observe that if μ is a binary measure then the Choquet integral always evaluates to one of its arguments, $Choq_\mu(V_1,\ldots, V_n) \in \{V_1,\ldots, V_n\}$.

It can be shown that the Choquet integral is an aggregation operator [2]. That is for any measure μ we have: 1) $Choq_\mu(0,\ldots, 0)=0$, 2) $Choq_\mu(1,\ldots, 1)=1$ and 3) $Choq_\mu(a_1,\ldots, a_n) \geq Choq_\mu(b_1,\ldots, b_n)$ when $a_i \geq b_i$ for all i. More specifically it is a mean-type aggregation operator, $Min_i[a_i] \leq Choq_\mu(a_1,\ldots, a_n) \leq Max_i[a_i]$. As all mean-type aggregation operators it is idempotent, i.e. if all $a_i=a$ then $Choq_\mu(a_1,\ldots, a_n)=a$.

One important implication of this fact that it is an aggregation operator is that we can use the Choquet integral to aggregate fuzzy measures to find more measures. Assume $M=\{\mu_1,\ldots, \mu_i,\ldots, \mu_n\}$ are a collection of measures on space X.