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On the relationship between symmetric maxitive, minitive, and modular aggregation operators

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ABSTRACT

In this paper the relationship between symmetric minitive, maxitive, and modular aggregation operators is considered. It is shown that the intersection between any two of the three discussed classes is the same. Moreover, the intersection is explicitly characterized.

It turns out that the intersection contains families of aggregation operators such as OWMax, OWMin, and many generalizations of the widely-known Hirsch's *h*-index, often applied in scientific quality control.

1. Introduction

Aggregation operators consist of functions used to combine multiple numeric values into a single one, in some way representative of the whole input. They may be applied in many areas of human activity, e.g. in statistics, engineering, operational research, quality control, image processing, pattern recognition, webometrics, and scientometrics. For example, in scientific quality control at an individual level, we are often interested in assessing authors of scholarly publications by means of the number of citations received by each of his/her papers or by using some other measures of their quality, see e.g. [11].

From now on let $\mathbb{I} = [a, b]$ denote any closed interval of the extended real line, $\overline{\mathbb{R}} = [-\infty, \infty]$. Note that in many practical applications we set $\mathbb{I} = [0, 1]$ or $\mathbb{I} = [0, \infty]$, cf. [9,1], respectively. The set of all vectors of arbitrary length with elements in $\mathbb{I}$, i.e. $\bigcup_{n=1}^{\infty} \mathbb{I}^n$, is denoted by $\mathbb{I}^{1,2,\ldots}$.

Let $\mathcal{E}(\mathbb{I})$ denote the set of all *aggregation operators* in $\mathbb{I}^{1,2,\ldots}$, i.e. $\mathcal{E}(\mathbb{I}) = \{\mathsf{F} : \mathbb{I}^{1,2,\ldots} \to \mathbb{I}\}$. Please note that the aggregation (averaging) functions, cf. [1,14,15,13], most commonly appearing in the literature, form a particular subclass of $\mathcal{E}(\mathbb{I})$. We require each such function F to be nondecreasing in each variable and to fulfill two boundary conditions: $\mathsf{F}(a, a, \ldots, a) = a$ and $\mathsf{F}(b, b, \ldots, b) = b$. Also observe that typically the aggregation (averaging) functions are considered for a fixed-length input vectors (for other approaches, see e.g. [3,4,8,12,19,21]).

In this paper, however, we focus our attention on aggregation operators that are only nondecreasing (in each variable) and, additionally, symmetric (i.e. which do not depend on the order of elements' presentation). The boundary conditions

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are omitted in our framework, as in some applications they are too restrictive [9,8]. Even though, note each aggregation operator F is a function into $\mathbb{I}$, therefore there is an implicit assumption that $\inf \mathsf{F} \geqslant a$, and $\sup \mathsf{F} \leqslant b$.

Definition 1. We say that $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ is *symmetric*, denoted $\mathsf{F} \in \mathcal{P}_{(\mathrm{sym})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \cong \mathbf{y} \Rightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \cong \mathbf{y}$ if and only if there exists a permutation σ of $[n] := \{1, 2, \ldots, n\}$ such that $\mathbf{x} = (y_{\sigma(1)}, \ldots, y_{\sigma(n)})$

Definition 2. We say that $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ is *nondecreasing*, denoted $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \leqslant \mathbf{y} \Rightarrow \mathsf{F}(\mathbf{x}) \leqslant \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \leqslant \mathbf{y}$ if and only if $(\forall i \in [n])\ x_i \leqslant y_i$.

In the theory of aggregation we are often interested in aggregation operators which fulfill a number of desirable properties. Among most basic ones we may find e.g. maxitivity, minitivity, additivity, see [13], or modularity [20,17]. In this paper we study the relationship between the symmetrized versions of these properties.

1.1. Notational convention

If not stated otherwise explicitly we assume that $n, m \in \mathbb{N}$. Arithmetic and lattice operations on vectors of the same length, e.g. $+, -, \vee$ (maximum), $\wedge$ (minimum), are always performed element-wise. Let $x_{(i)}$ denote the ith order statistic of $\mathbf{x} \in \mathbb{I}^n$.

For each $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$, $(\mathbf{x}, \mathbf{y})$ denotes the concatenation of the vectors, i.e. $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{I}^{n+m}$. A vector $(x, x, \ldots, x) \in \mathbb{I}^n$ is denoted briefly by $(n * x)$.

If $\mathsf{f}, \mathsf{g} : \mathbb{I} \to \mathbb{I}$, then $\mathsf{f} \preceq \mathsf{g}$ (g dominates f) if and only if $(\forall x \in \mathbb{I})\ \mathsf{f}(x) \leqslant \mathsf{g}(x)$.

Additionally, $\mathbf{1}$ denotes the indicator function.

In the next section we present and characterize three very interesting classes of symmetric aggregation operators, with which we are concerned in this paper.

2. Symmetric maxitive, minitive, and modular aggregation operators

2.1. Definitions

Let us first recall the notion of a triangle of functions [10,7]:

Definition 3. A *triangle of functions* is a sequence $\triangle = (\mathsf{f}_{i,n} : \mathbb{I} \to \mathbb{I})_{i \in [n], n \in \mathbb{N}}$.

Note that such an object is similar to a triangle of coefficients, $(c_{i,n} \in \overline{\mathbb{R}})_{i \in [n], n \in \mathbb{N}}$, considered, e.g. in [4,8,19].

Quasi-S- and quasi-L-statistics were introduced in [10].

Definition 4. A *quasi-S-statistic* generated by a triangle of functions $\triangle$ is a function $\mathsf{qS}_\triangle \in \mathcal{E}(\mathbb{I})$ defined for any $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$ as $\mathsf{qS}_\triangle(\mathbf{x}) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)})$.

Quasi-S-statistics generalize the well-known OWMax operators [5] for which we have $\mathsf{f}_{i,n}(x) = x \wedge c_{i,n}, c_{i,n} \in \mathbb{I}$, and $(\forall n)$ $\bigvee_{i=1}^n c_{i,n} = b$.

Definition 5. Let $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ be a triangle of functions such that $(\forall n)\ \sum_{i=1}^n \inf \mathsf{f}_{i,n} \geqslant a$, and $\sum_{i=1}^n \sup \mathsf{f}_{i,n} \leqslant b$. Then the *quasi-L-statistic* generated by $\triangle$ is a function $\mathsf{qL}_\triangle \in \mathcal{E}(\mathbb{I})$ defined for any $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$ as $\mathsf{qL}_\triangle(\mathbf{x}) = \sum_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)})$.

Note that, e.g. the condition $(\forall n)\ \sum_{i=1}^n \inf \mathsf{f}_{i,n} \geqslant a$ is important for $a < 0$. The class of quasi-L-statistics includes the OWA operators [25] (for $0 \in \mathbb{I}$) for which it holds $\mathsf{f}_{i,n}(x) = c_{i,n}x, c_{i,n} \in [0, 1]$, $(\forall n)\ \sum_{i=1}^n c_{i,n} = 1$, and the OMA operators [20] with $(\forall n)$ $\sum_{i=1}^n \mathsf{f}_{i,n} = \mathrm{id}$.

Let us introduce another interesting class of aggregation operators.

Definition 6. A *quasi-I-statistic* generated by a triangle of functions $\triangle$ is an aggregation operator $\mathsf{qI}_\triangle$, for which we have $\mathsf{qI}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)})$, where $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$.

This class of functions generalizes the OWMin operators [5], for which we have $\mathsf{f}_{i,n}(x) = x \vee c_{i,n}$, where $c_{i,n} \in \mathbb{I}$, and $(\forall n)$ $\bigwedge_{i=1}^n c_{i,n} = a$. However, observe that for each OWMax operator there exists an equivalent OWMin operator, and inversely [13].

The name L-statistics (linear combination of order statistics or, sometimes, linear combination of a function of order statistics) probably first appeared in [2] in the field of probability. Moreover, please note that $\vee$ and $\wedge$ denotes the maximum (*S*upremum) and, respectively, the minimum (*I*nfimum) operator, hence the other names.

It may easily be shown that the restriction of quasi-S-statistics and quasi-I-statistics to $\mathbb{I}^n$ (for any n) generalizes Sugeno integrals (cf. [13]) of $\mathbf{x} \in \mathbb{I}^n$ with respect to any monotonic symmetric set function $\xi : 2^{[n]} \to \mathbb{I}$. Additionally, it should be noted