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Interval representations, Łukasiewicz implicators and Smets–Magrez axioms

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ABSTRACT

The Smets–Magrez axiomatic is usually used to define the class of fuzzy continuous implications which are both S and R-implications (Łukasiewicz implications). Another approach is the construction of such class starting from a basic implication and applying automorphisms. Literature has shown that there is a harmony between those approaches, however in this paper we show that the extension of the Łukasiewicz implication defined on [0, 1] for interval values cannot be applied in a direct way.

We show that the harmony between the Smets–Magrez axiomatic approach and the one that comes from the generation by automorphisms is not preserved when such extension is done. One of the main consequences lies on the fact that the automorphism approach induces the loss of R-implications from the resulting class of implicators. More precisely, we show that the interval version of such approaches produce two disjunct classes of implicators, meaning that, unlike the usual case, the choice of the respective approach is an important step.

1. Introduction

In 1987, Smets and Magrez [30] provided an axiomatic which characterizes implications defined on [0,1] which are continuous, S- and R-implications. After that, it was shown that the Łukasiewicz implication (up to a conjugation) is the only continuous fuzzy implication which is an S-implication and an R-implication (c.f. [16,17,21]). Baczyňski [2] also demonstrated that the original axiomatic can be reduced to continuity, exchange and confinement axioms. Therefore, Smets–Magrez axiomatic characterizes both the intersection between the class of continuous S-implications with the class of continuous R-implications and the conjugates of the Łukasiewicz implication. This approach contains two methodologies which are compatible in the scenario of [0, 1]: (1) The definition of the class of implications from Smets–Magrez axiomatic; and (2) The generation of such class from the Łukasiewicz implication via automorphisms. However, when such methodologies are transported to the interval scenario we observe an incompatibility; namely: The definition of the class of interval implication through the axiomatic produces a class of implicators which is disjunct from that produced by interval automorphisms from any interval implication (in particular from the best interval representation of standard Łukasiewicz implication).

Cornelis et al. [10] proposed an extension of the Smets–Magrez axiomatic for Intuitionistic Implication (which can be directly translated to Interval Implications [13]). It reveals the following characteristics: (1.1) All Smets–Magrez axioms are satisfied; (1.2) The axiomatic specifies the intersection of the class of S- with R-implications; (1.3) Every resulting

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implicator is not i-representable[1] (e.g. (S,N)-implications in [23] and R-implications in [10]). On the other hand, this paper, applies methodology (2), since we generate the class of interval implicators which are obtained by interval autormorphisms from the best interval representation of the standard Łukasiewicz implication. This reveals the following characteristics: (2.1) The resulting implicators do not fully satisfy the Smets–Magrez axiomatic, they fail to satisfy the Confinement axiom. In this context, we propose a weakening of Confinement (Theorem 4.5); and (2.2) Every resulting implicator is i-representable. Therefore, the harmony between the two approaches could not be inherited from implications on [0,1] to interval implications.

Cornelis et al. [10] presented the open problem about the lack of knowledge about the existence of an intuitionistic Łukasiewicz implicator i.e. a conjugate of the Łukasiewicz intuitionistic implication $I((x_1,x_2),(y_1,y_2)) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$,[2] which does not preserve diagonal elements (degenerate intervals). The positive answer together with characteristic (1.3) would imply the impossibility to generate the whole class of those intuitionistic Łukasiewicz implicators from a basic implication through automorphisms. On the other hand, the negative answer would imply the introduction of uncertainty on values which are not uncertain.[3]

The paper is structured in the following way: Section 2 provides some insights about interval representation of connectives. Section 3 provides some background about Łukasiewicz implicators on {0,1}. Finally, Section 4 develops the investigations proposed here.

2. Interval representation of fuzzy connectives

Some previous papers provide interval extensions for some fuzzy connectives (e.g. [6,4,7,8,14,15,28,18,27,9,26]) by considering both correctness (accuracy) and optimality aspects of interval methods [22,29]. In this paper, following the same approach, we will investigate Łukasiewicz fuzzy implications and their related properties.

Consider the real unit interval $U = [0,1] \subseteq \mathbb{R}$ and the set $\mathbb{U} = \{[a,b] \mid 0 \leqslant a \leqslant b \leqslant 1\}$ of subintervals of U. The *left* and *right projections* of an interval $[a,b] \in \mathbb{U}$ are given by the functions $l, r : \mathbb{U} \to U$, respectively defined by

$$l([a,b]) = a \text{ and } r([a,b]) = b. \tag{1}$$

For a given interval, $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also respectively denoted by $\underline{X}$ and $\overline{X}$.

In what follows we consider two important interval partial ordering:

(i) The *product* order (also called component-wise order or *Kulisch–Miranker* order): for all $X, Y \in \mathbb{U}$,

$$X \leqslant Y \iff \underline{X} \leqslant \underline{Y} \wedge \overline{X} \leqslant \overline{Y}; \tag{2}$$

(ii) The *inclusion* order: for all $X, Y \in \mathbb{U}$,

$$X \subseteq Y \iff \underline{X} \geqslant \underline{Y} \wedge \overline{X} \leqslant \overline{Y}. \tag{3}$$

Since any partial order can be extended component-wisely, (i) and (ii) can be naturally extended to $\mathbb{U}^n$ in the following way:

$$\overrightarrow{X} \leqslant \overrightarrow{Y} \iff X_1 \leqslant Y_1 \wedge \cdots \wedge X_n \leqslant Y_n. \tag{4}$$

and

$$\overrightarrow{X} \subseteq \vec{Y} \iff X_1 \subseteq Y_1 \wedge \cdots \wedge X_n \subseteq Y_n \tag{5}$$

for any $\overrightarrow{X} = (X_1, \ldots, X_n), \overrightarrow{Y} = (Y_1, \ldots, Y_n) \in \mathbb{U}^n$.

The notion of interval correctness plays an important role in numerical computations; it means that a correct interval method can always guarantee that: $x \in X \Rightarrow f(x) \in F(X)$; where F is the interval method which calculates a real function f. Santiago et al. [29] explored how this notion of interval correctness relates to the computing relation with the Euclidean topology and the several viewpoints of intervals. In this paper, the notion of correctness is named *Interval Representation*, with the aim to formalize the authors' viewpoint that *interval methods are representations of (a set of) punctual methods*. In the following we reproduce such definition, but instead of $\mathbb{R}$, we consider U.

Definition 2.1. [29] An interval $X \in \mathbb{U}$ is a *representation* of any real number $\alpha \in X$. Considering two interval representations X and Y for a real number α, X is said to be an *interval representation* of α *better* than Y, if $X \subseteq Y$. This notion can also be naturally extended for n-tuples of intervals.

Definition 2.2. [29] A function $F : \mathbb{U}^n \to \mathbb{U}$ is said to be an *interval representation* of a real function $f : U^n \to U$ if, for each $\overrightarrow{X} \in \mathbb{U}^n$ and $\vec{x} \in \overrightarrow{X}$, $f(\vec{x}) \in F(\overrightarrow{X})$. F is also said to be *correct* with respect to f. An interval function $F : \mathbb{U}^n \to \mathbb{U}$ is said to be an *interval representation* of a real function $f : U^n \to U$ *better* than an interval function $G : \mathbb{U}^n \to \mathbb{U}$, if $F(\overrightarrow{X}) \subseteq G(\overrightarrow{X})$, for each $\overrightarrow{X} \in \mathbb{U}^n$.

[1] in the sense of Deschrijver et al. [11]

[2] Where the interval counterpart are the conjugates of the implication $I(X,Y) = [\min(1, \underline{Y} + 1 - \underline{X}, 1 - \overline{X} + \overline{Y}), \min(1, 1 - \underline{X} + \overline{Y})]$; for $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$.

[3] Because of enlargements of some degenerate (exact) values.