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On clarifying some definitions and notations used for type-2 fuzzy sets as well as some recommended changes

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ABSTRACT

This paper is about some important changes in type-2 fuzzy set (T2 FS) definitions and notations that have occurred during the past 16 years. It summarizes the evolution of how the primary membership (J_x) has been used in both the mathematical descriptions of a T2 FS and its footprint of uncertainty (FOU); discusses notational problems associated with the secondary membership function; explains why and when one should distinguish between the FOU and the domain of uncertainty (DOU); explains why no errors have been introduced into T2 FS computations because of notation; and, it provides recommendations notational changes that can be used by all authors.

1. Introduction

In 1975 Zadeh introduced type-2 fuzzy sets[1] [31]; however, not much was published about them[2] prior to 1999 (e.g., [3–5,25,26,28]), when Mendel and his students began to publish many articles about them. Currently, thousands of articles have been published about them (see Fig. 1 in [30]; it's a plot of the number of Google Scholar items about type-2 (T2) fuzzy logic for the years 1995 through 2013). During the past 17 years some of the definitions and notations about type-2 fuzzy sets (T2 FSs) have changed, for the better. The purposes of this paper are to provide a history of the changes and make recommendations so that new authors to the T2 field will not have to wade through the large numbers of T2 papers in order to do this themselves, and so that they (as well as present authors) can use the definitions and notations that are recommended herein.

Ref. [1] is a very important reference about T2 definitions and notations. Its Table I defines a fuzzy set term, and then provides its fuzzy set notation, translated terminology and standard mathematical notation. Although the authors of [1] would prefer T2 authors to use the standard mathematical notation, this has not happened. We have found that almost all

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[1] As noted in [16, footnote 2]: "In the early days of type-2 fuzzy sets and systems, the phrases 'type-2 fuzzy set' or 'type-2 fuzzy system' were used in an all-inclusive way, meaning any kind of type-2 fuzzy set or system. During the past 15 years most of the attention has been given to interval type-2 fuzzy sets and systems. It is only within the past 5 years or so that there has been a return to more general type-2 fuzzy sets and systems, and, to distinguish them from the more specialized interval type-2 fuzzy sets and systems, the term 'general' is being used. In essence, type-2 fuzzy sets and systems now consist of the union of interval and general type-2 fuzzy sets and systems."

[2] See [12, Section 1.6] for a brief history of the publications prior to 2001.

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0020-0255/

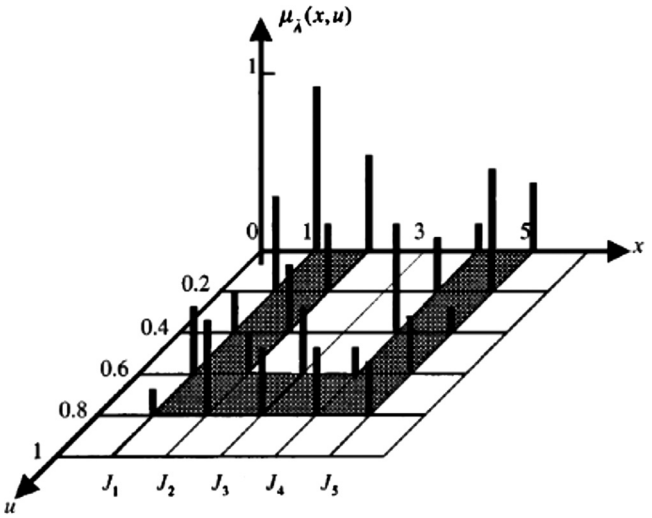

Fig. 1. Example of a type-2 membership function. The shaded area is the FOU [20].

authors of T2 papers, since the appearance of [1], are either unaware of it, are using only some of it (i.e., they are using a mixture of the standard mathematical notation and fuzzy set notation), or are ignoring it. All of this may be due to the very large literature that now exists for T2, the arguable reluctance of new authors to scour the large literature (many authors refer to [12], which only uses the "fuzzy set notation"), or the arguable reluctance of authors who have published T2 papers prior to 2010 to adopt changes (i.e., they believe it is important to use the same notation used in the past so that readers can easily connect to the past). Our recommended notation changes will try to honor both the past as well as the standard mathematical notation of [1]. Moreover, this paper clarifies and slightly modifies several definitions in T2 fuzzy set theory.

The organization of the rest of this paper is: Section 2 provides background material that is needed in the rest of the paper; Section 3 summarizes the evolution of how J_x (the primary membership of a T2 FS) has been used in both the mathematical descriptions of a T2 FS and its footprint of uncertainty (FOU), two interpretations for J_x (one correct and the other incorrect), and some recommendations; Section 4 describes some notations for and associated notational issues about the secondary membership function of a T2 FS; Section 5 explains the difference between the FOU and the Domain of Uncertainty (DOU) of a T2 FS; Section 6 explains why no errors have been introduced into T2 FS and fuzzy logic systems (FLSs) mathematical results, even though some T2 FS definitions and notations have been poor; and, Section 7 draws conclusions and summarizes our recommended modifications.

2. Background

Because [12] is the most heavily referenced publication about T2, and is also the one that we observe most new T2 authors are referring to, we begin by providing some definitions that are in it. Note that, strictly speaking, the ∀ symbol should be removed everywhere that it appears in Definitions 3-1, 3-2, 3-3 and 3-8; it is shown in those definitions below because it appears in those definitions in [12]. Note, also, that we are only including all of these definitions, with their problematic notation, as a starting point for the rest of this paper.

In these definitions, X is the universe of discourse for the primary variable (x)—a set—, and as stated in [1]: "While the structure of universe is rarely used explicitly to constrain membership functions, the literature is dominated by examples in which the universe is a subset of the real line." Additionally, the universe of discourse (which is not mentioned explicity in the definitions), U, for the secondary variable u, is always assumed to be $[0, 1]$.

Definition 3-1. *A* T2 FS *denoted* $\tilde{A}$*, is characterized by a* type-2 membership function (*T2 MF*) $\mu_{\tilde{A}}(x,u)$*, where* $x \in X$ *and* $\forall u \in J_x \subseteq [0,1]$*, i.e.,*

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \tag{1}$$

in which $0 \le \mu_{\tilde{A}}(x,u) \le 1$. $\tilde{A}$ *can also be expressed as*

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u) \quad J_x \subseteq [0,1] \tag{2}$$

where $\int\int$ *denotes the union over all admissible x and u.*

Note that x and u are called the *primary* and *secondary* variables of $\tilde{A}$, respectively.