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Convergence rate on periodic gossiping[☆]

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ABSTRACT

In a sensor network in which each sensor controls a real-valued state, the goal of a distributed averaging problem is to compute the global average in a decentralized way, which is the average of all sensors' initial state values across the entire network. A T-periodic gossiping protocol can solve such a problem, which stipulates that each agent must gossip with each of its neighbors exactly once every T time unit. The convergence rate of a T-periodic gossiping protocol is determined by the magnitude of the second largest eigenvalue of the stochastic matrix associated with the gossip sequence occurring over one period. An interesting result is that when the allowable gossip graph is a tree, the convergence rate is independent of gossip orders within one period. This paper will prove this result by developing several properties of doubly stochastic matrices. The properties derived also can be used in analyzing convergence rate problems of other periodic gossip protocols.

1. Introduction

Recently an increasing interest has been given to sensor networks and distributed control [6,12,14,17,23], in which each sensor node controls a state initialized by a real-valued measurement and is able to communicate with certain other sensor nodes called its "*neighbors*". Since the measurements in practice are usually with zero-mean noise, distributed averaging algorithms become more and more important, which compute the average of the initial state values of all nodes across the network by only communications among neighbors. Consensus algorithms have been studied in [3–5,7,11,13,26,27] with applications into developing a distributed algorithm for solving linear algebraic equations [20–22]. Different from consensus algorithms, distributed averaging algorithms not only drive all the states to converge to be the same steady state but also require the steady state value to be the average of initial states of all nodes across the network [19].

One type of distributed averaging algorithms is the Metropolis Algorithm [28], in which each node is usually assumed to be able to broadcast and performs updates according to all the states of its neighbors at each iteration. Convergence of the Metropolis Algorithm requires each node to know the degrees of all its neighbors to compute the Metropolis weights. Another popular type of distributed algorithms are gossiping algorithms [2,8,16,24,25], in which pairs of nodes *gossip* at

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each iteration by updating their state values to the average of their previous ones. Thus in gossiping algorithm each node is only required to communicate with one neighbor node instead of broadcasting. In [2] randomized gossiping algorithms are studied, in which a randomly chosen node is activated at each iteration and randomly chooses a neighbor node to gossip with. There exists the case that the same pair of nodes gossip again right after they finish their previous gossip, which may lead to a slow convergence. To overcome this problem and also enable more than one disjoint pairs of nodes to gossip at each time, deterministic gossiping are proposed in [16]. To further speed up the convergence, acceleration gossip algorithms were developed methods in [15,18].

In this paper we mainly focus on a deterministic periodic gossiping algorithm, in which one pair of nodes gossip only once in one period. The convergence rate of such a periodic gossiping protocol is determined by the magnitude of the second largest eigenvalue of a matrix associated with the gossip sequence occurring over one period, which generally depends on the order of the gossips within one period. However, when the so-called "allowable gossip graph", in which there is an edge between a pair of nodes if and only if these two nodes are neighbors, is a tree, the convergence rate does not change with respect to the order of gossips. This interesting result was first proved in [1] based on an important concept " transfer function" in control field. The aim of this paper is to provide a more quantitative analysis for this result. We will look directly at the properties of a class of doubly stochastic matrices associated with the sequence of gossips occurring over a given period, explore their nice properties and reprove the invariant property of convergence rate for the periodic gossiping algorithm in a different way. The properties derived in this paper bridge the connection of a tree graph and the associated gossip matrices, which as well have a potential to be applied into analyzing other convergence rate problems of periodic gossip algorithms.

The rest of the paper is organized as follows: Section 2 defines the periodic gossip sequence and presents the convergence rate invariance result. In Section 3, we derive several properties of a class of doubly stochastic matrices, based on which we give the proof of the convergence rate result in Section 4. In Section 5, several numerical simulations on typical tree graphs are provided to show the effectiveness of the convergence rate invariance result. Concluding remarks and future work are given in Section 6.

2. Periodic gossiping and main result

In this section we will first define the periodic gossip sequence and analyze its convergence, based on which the convergence rate invariance result of this paper will be given.

2.1. Periodic gossip sequence

Consider a sensor network with n nodes, in which each node controls a real-value state $x_i(t) \in \mathbb{R}$ called a *gossip variable*. Suppose each node is able to communicate with other certain nodes called its *neighbors*. Let time be discrete, that is, $t \in \{0, 1, 2, \dots\}$. At each time instant t, one pair of neighbor nodes i and j *gossip* by updating their gossip variables to be the average of their previous values while all other agents' gossip variables remains unchanged. That is, if i and j gossip at time t, one has

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}(x_i(t)+x_j(t)), \\ x_j(t+1) &= \frac{1}{2}(x_i(t)+x_j(t)), \\ x_k(t+1) &= x_k(t), \quad k \neq i, j. \end{aligned} \tag{1}$$

Let $\mathcal{N}_i$ denote the set of all neighbors of node i in the n-node sensor network. The neighbor relationship can be conveniently characterized by an undirected graph $\mathbb{A} = \{\mathcal{V}, \mathcal{E}\}$, in which $\mathcal{V} = \{1, 2, \dots, n\}$ is the node set and $(i, j) \in \mathcal{E}$ is the edge set if and only if node i and j are neighbors. The graph $\mathbb{A}$ is also called the *allowable gossip graph* since a gossip is allowable to take place between the node i and j if and only if $(i, j) \in \mathcal{E}$.

A *gossip sequence* is defined as an ordered sequence of edges for a given allowable gossip graph $\mathbb{A}$ in which each edge appears at most once. The gossip sequence is *complete* if $\mathbb{A}$ is connected and it is called *minimally complete* if $\mathbb{A}$ is a tree. An infinite repetitive gossip sequence is called *periodic* with period T if each gossip in the sequence occurs once in every T time unit. Such a sequence is called *periodically complete* and *periodically minimally complete* if any its subsequence over one period is complete and minimally complete, respectively.

2.2. Convergence rate of a periodic gossip sequence

To model the above updating equations (1) in state-space, we define an $n \times n$ *primitive gossip matrix* P_{ij} with entries $p_{ii} = p_{ij} = p_{ji} = p_{jj} = \frac{1}{2}$, $p_{kk} = 1, k \neq i, j$ and all other entries equal zero. Let $x(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^{\mathrm{T}}$. Then the evolution in the state-space when i and j gossip at time t can be described by the following discrete time linear system model

$$x(t+1) = P_{ij}x(t), \quad t \geq 0. \tag{2}$$