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A generalized distance based on a generalized triangle inequality

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ABSTRACT

In this paper we present a new generalization of the mathematical notion of distance. It is based on the abstraction of the codomain of the distance function. The resulting functions must satisfy a generalized triangular inequality, which depends only on the order structure of the valuation space, i.e a monoid structure is not required. This type of functions will be called **i-Distances** (i-metrics, i-quasi-metric, etc.). We show that they generate a topology in a very natural way based on open balls. This paper generalizes (Santanaand Santiago, 2013) which has been successfully applied in the field of Clustering Algorithms (see Silva et al., 2014, 2015). An example in the field of Interval Mathematics is also investigated. The resulting topology is Hausdorf and regular but non-metrizable, what means that it cannot be generated by an usual metric.

1. Introduction

The mathematical representation of the concept of distance on a set M are the functions of type $d : M \times M \longrightarrow \mathbb{R}$ which satisfy some conditions. The most used distance function is called *metric* and the conditions are:

1. $d(x, y) \geq 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) = 0$ if, and only if, $x = y$;
4. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangular inequality*).

One way to generalize the notion of distance is by modifying the codomain of the distance function, namely, the distance between two elements of the set M is not necessarily a real number. This kind of generalization is not new. The first one was proposed by Menger in 1942 [12] and was called **Statistical Metric**. After that, another kind of statistical metric was proposed [19]. In both cases, the distance between two elements is a *probability distribution function*. Kramosil *and Michelek* [10], adapted the first proposal of statistical metric to the fuzzy set context providing the first concept of **Fuzzy Metric Spaces**. This concept was modified in [4] and [7] and continued to be called fuzzy metric. Another generalization founded in the literature is the notion of **generalized metrics** [8]. This last one was proposed to obtain a fixed point theorem for multivalued applications on disjunctive logic programs. In this context, the codomain of the distance is an ordered abelian monoid and the conditions are the same as in usual metric. There is yet another generalization called **Continuity Spaces**, whose the

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first version was proposed in [9] and the second in [3]. The concepts of statistical metrics and fuzzy metrics are linked with the theories of probability and fuzzy logic, whereas that of generalized metrics and continuity spaces are more general.

Like generalized metrics and continuity spaces, the goal of the present work is to provide an abstract notion which is not connected with an specific context, although the initial idea was to propose a generalization able to satisfactorily capture the idea of an interval metric and interval representation (see [13,14,18]). Another goal was to provide a kind of distance able to generate a topology in a natural way, i.e., based on open balls. We prove in this paper that these goals are obtained with our generalized metric, which we call **i-metrics**.

2. i-Distance valuation

In this first section, we specify the codomains of the new distances. This section requires a few knowledge in order theory and domain theory. Some of the notions presented here are new, like **semi-auxiliary relation**, **separable smallest element** and **IDV** (short for **i-distance valuation**).

Definition 1. Let $\leq$ be a partial order on A (in this case $\langle A, \leq\rangle$ is called a poset). A binary relation R on A is a **semi-auxiliary relation to $\leq$** whenever:

1. $aRb \Rightarrow a \leq b$;
2. If $a \leq b$, bRc and $c \leq d$, then aRd.

This definition is very similar to that of **auxiliary relation** in Domain Theory [5]. The reason to provide this new concept is to abstract the notion of strict order $<$, which is fundamental for open balls, but cannot be directly abstracted to the notion of auxiliary relation, since for posets with smallest element $\perp$, like $\langle \mathbb{R}^+, \leq, 0\rangle$, it will not be true that $\perp < \perp$.

Proposition 1. *If $\langle A, \leq\rangle$ is a poset, then the strict relation $a < b \Leftrightarrow (a \leq b) \wedge (a \neq b)$ is a semi-auxiliary relation to $\leq$.*

Proof. The first condition is immediate. To prove the second, suppose that $a \leq b$, $b < c$ and $c \leq d$. Since $b < c$, we have $b \leq c$ and, by the transitivity of $\leq$, we have $a \leq d$. It only remains to verify that $a \neq d$. Suppose that $a = d$. Thus $b = c$, which contradicts the hypothesis $b < c$. Thus, we must have $a \neq d$, so $a < d$. □

Another semi-auxiliary relation for partial orders is the way-below relation [5]:

Definition 2 (Way-below)**.** Consider a poset $\langle A, \leq\rangle$. We say that the element x is **way-below** an element y, which is denoted by $x \ll y$, if for every directed set $D \subseteq A$ with supremum such that $y \leq \sup D$, then exists $d \in D$ such that $x \leq d$. In the case of posets with smallest element $\perp$, we say that x is **strictly way-below** y, $x \ll^* y$, whenever $x \ll y$ and $y \neq \perp$.

Proposition 2. *If $\langle A, \leq\rangle$ is a poset, then the way-below relation $\ll$ is a semi-auxiliary relation to $\leq$.*

Proof. Since every auxiliary relation is a semi-auxiliary relation and $\ll$ is an auxiliary relation (see [5]), the proposition holds. □

Observation 1. If $\langle A, \leq, \perp\rangle$ is a poset with smallest element $\perp$ and $\ll$ is the way-below relation to $\leq$, then $\perp \ll x$, for all $x \in A$ (see [5]), particularly $\perp \ll \perp$. This will bring about problems for the notion of open balls, for this reason we introduced the notion of strict way-below relation.

Proposition 3. *If $\langle A, \leq\rangle$ is a poset, then $\ll^*$ is a semi-auxiliary relation to $\leq$.*

Proof. Immediate. □

Proposition 4. *If $\langle A, \leq\rangle$ is a poset, then every semi-auxiliary relation to $\leq$ is transitive.*

Proof. Let R be a semi-auxiliary relation to $\leq$ and suppose that aRb and bRc. Thus, we have $a \leq b$, bRc and $c \leq c$, so, from the second condition, it follows that aRc. □

Below, we introduce another concept that will be important for the valuation spaces of i-distances.

Definition 3. A poset with smallest element and a semi-auxiliary relation R, $\langle A, \leq, R, \perp\rangle$, is said to have **separable smallest element**, whenever A is **d-directed**[1] and for every pair of elements $a, b \in A$, with $\perp Ra$ and $\perp Rb$, there is a lower bound c for $\{a, b\}$ such that $\perp Rc$.

Example 1. Consider $\mathbb{N}^* = \{1, 2, \ldots, \}$, the partial order $a \leq_d b \Leftrightarrow a|b$ and it's strict relation $<$. The smallest element of $\langle \mathbb{N}^*, \leq_d\rangle$ is 1 and for $a, b \in \mathbb{N}^*$, it is easy to see that $gcd(a, b)$(greatest commom divisor) is a lower bound for $\{a, b\}$. Note that $gcd(2, 3) = 1$, so the unique lower bound of $\{2, 3\}$ is 1. Thus, this poset has no separable smallest element.

On the other hand, if $\leq$ is a total order, then $\langle A, \leq, <, \perp\rangle$ has separable smallest element.

Finally, we can define the structure which will be used as codomain of i-distances:

[1] i.e. every subset of A with two elements has lower bound.