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On the Bernoulli automorphism of reversible linear cellular automata

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ABSTRACT

This investigation studies the ergodic properties of reversible linear cellular automata over $\mathbb{Z}_m$ for $m \in \mathbb{N}$. We show that a reversible linear cellular automaton is either a Bernoulli automorphism or non-ergodic. This gives an affirmative answer to an open problem proposed by Pivato [20] for the case of reversible linear cellular automata.

1. Introduction

Motivated by biological applications, John von Neumann introduced cellular automaton (CA) in the late 1940s. The main goal was to design self-replicating artificial systems that are also computationally universal and are analogous to human brain. That is, CA is designed as a computing device in which the memory and the processing units are not separated from each other; such a device is massively parallel and capable of repairing and building itself given the necessary raw material.

CA has been systematically studied by Hedlund from a purely mathematical point of view [9]. For the past few decades, studying CA from the viewpoint of the ergodic theory has received lots of attention [1,2,5,6,10,14,21,22]. Pivato characterized the invariant measures of bipermutative right-sided, nearest neighbor cellular automata [19]. Moreover, Pivato and Yassawi introduced the concepts of harmonic mixing for measures and diffusion for a linear CA, and developed broad sufficient conditions for the convergence of the limit measures [21,22]. Sablik demonstrated the measure rigidity and directional dynamics for CA [23,24]. Host *et al.* studied the role of uniform Bernoulli measure in the dynamics of cellular automata of algebraic origin [10]. Furthermore, some sufficient conditions for whether a one-dimensional permutative CA is strong mixing, k-mixing, or Bernoulli automorphic were independently revealed by Kleveland and Shereshevsky [14,25,26]. Recently, one-sided expansive invertible cellular automata and two-sided expansive permutation cellular automata have been demonstrated to be strong mixing (see [3,4,12,13,17]).

Almost all the results about ergodic properties are for one-dimensional (mostly permutative) CA and for the uniform measure. It is natural to ask the following question:

Problem 1 (See [20]). Can mixing and ergodicity be obtained for non-permutative CA and/or non-uniform measures? What about multidimensional CA?

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Theorem 2.5 and Corollary 2.4 indicate that an invertible linear CA is either Bernoulli automorphic or non-ergodic for the uniform Bernoulli measure. In [6], Cattaneo *et al.* addressed a necessary and sufficient condition for the ergodicity of linear CA. Corollary 2.4 reveals a concise condition for the ergodicity of invertible CA. The result remains true for those measures satisfying some conditions (see Remark 6.1). The methodology can be extended to investigate the multidimensional invertible linear CA, and even possibly the non-permutative cases. Related works are under preparation.

The rest of this paper is organized as follows. Section 2 states the main results and some preliminaries. The proofs are postponed to Sections 4 and 5 while the key ideas are revealed via some examples in Section 3. Discussion and further works are addressed in Section 6.

2. Statement of main results

Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ be the ring of the integers modulo m, where $m \geq 2$, and let $\mathbb{Z}_m^{\mathbb{Z}}$ be the space of all doubly-infinite sequences $x = (x_n)_{n=-\infty}^{\infty} \in \mathbb{Z}_m^{\mathbb{Z}}$, equipped with the product of the Tychonoff topology. Then the shift $\sigma : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(\sigma x)_i = x_{i+1}$ is a homeomorphism of the compact metric space $\mathbb{Z}_m^{\mathbb{Z}}$. A one-dimensional cellular automaton is a continuous map $T_f : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(T_f x)_i = f(x_{i+l}, \ldots, x_{i+r})$, where $l, r \in \mathbb{Z}$ and $f : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ is a given local rule or map. A local rule f is said to be *linear* if it can be written as

$$f(x_l, \ldots, x_r) = \Sigma_{i=l}^{r} \lambda_i x_i \pmod{m}, \quad l, r \in \mathbb{Z},$$

where at least one of $\lambda_l, \ldots, \lambda_r$ is nonzero in $\mathbb{Z}_m$ [8,16].

A local rule f is said to be *permutative* in x_j (or, at the index j) if for any given finite sequence

$$(\bar{x}_l, \ldots, \bar{x}_{j-1}, \bar{x}_{j+1}, \ldots, \bar{x}_r) \in \mathbb{Z}_m^{r-l}$$

we have

$$\{f(\bar{x}_l, \ldots, \bar{x}_{j-1}, x_j, \bar{x}_{j+1}, \ldots, \bar{x}_r) : x_j \in \mathbb{Z}_m\} = \mathbb{Z}_m.$$

The notion of permutative cellular automata was first introduced by Hedlund [9]. A linear local rule f is permutative at the index j if and only if $\gcd(\lambda_j, m) = 1$, where $\gcd(p, q)$ denotes the greatest common divisor of p and q.

For every linear local rule $f(x_l, \ldots, x_r) = \Sigma_{i=l}^{r} \lambda_i x_i \pmod{m}$, there associates a formal power series $F(X) = \Sigma_{i=l}^{r} \lambda_i X^{-i}$. Let T_f be the cellular automaton defined by the local rule f. Ito et al. characterize the necessary and sufficient condition for the invertibility of T_f.

Theorem 2.1 (See [11]). *T_f is invertible if and only if for each prime factor $p|m$ there exists a unique j_p such that f is permutative at the index j_p.*

For the case where $m = p^k$ for some prime number p and $k \in \mathbb{N}$, it is immediate that T_f is invertible if and only if there exists a unique j such that $\gcd(\lambda_j, p) = 1$. Manzini and Margara demonstrated that the corresponding formal power series $F(X)$ is invertible in $\mathbb{Z}_m[[X, X^{-1}]]$.

Theorem 2.2 (See [15]). *Suppose $m = p^k$ and T_f is invertible. Write $F(X) = \lambda_{j_p} X^{-j_p} + pH(X)$. Let $\widetilde{H}(X) = -\lambda_{j_p} X^{j_p} H(X)$. Then*

$$F^{-1}(X) = \lambda_{j_p}^{-1} X^{j_p} (1 + p\widetilde{H}(X) + \cdots + p^{k-1} \widetilde{H}^{k-1}(X)),$$

where $\lambda_{j_p}^{-1}$ is the inverse element of λ_{j_p} in $\mathbb{Z}_m$.

Let T: $X \to X$ be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$. T is called *strong mixing* if

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$$

for any $A, B \in \mathcal{B}$. Furthermore, T is called *k-mixing* if for every given $\{A_i\}_{i=0}^{k} \subset \mathcal{B}$,

$$\lim_{n_1, n_2, \ldots, n_k \to \infty} \mu(A_0 \cap T^{-n_1} A_1 \cap \cdots \cap T^{-(n_1 + \cdots + n_k)} A_k) = \mu(A_0)\mu(A_1) \cdots \mu(A_k).$$

Evidently, strong mixing is 1-mixing.

It is known that every surjective cellular automaton preserves the uniform Bernoulli measure (cf. [7,14,25] for instance). For the rest of this paper, μ refers to the uniform Bernoulli measure unless stated otherwise. Kleveland [14] and Shereshevsky [25,26] have proved that T_f is strong mixing if $r < 0$ (resp. $l > 0$) and f is left permutative (resp. right permutative); some of these cellular automata are even k-mixing. Recently, one-sided expansive invertible cellular automata and two-sided expansive permutation cellular automata have been demonstrated to be strong mixing (see [3,4,12,13,17]). Theorem 2.3 addresses the necessary and sufficient condition for whether an invertible linear cellular automaton is strong mixing, which extends the previous results and characterizes the strong mixing property of invertible linear cellular automata completely.

Theorem 2.3. *An invertible linear cellular automaton T_f is strong mixing with respect to the uniform Bernoulli measure if and only if $j_p \neq 0$ for every prime factor p of m.*