Information Sciences 266 (2014) 148–159

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Fuzzy identities with application to fuzzy semigroups

Branka Budimirović [a], Vjekoslav Budimirović [a], Branimir Šešelja [b], Andreja Tepavčević [b,*]

[a] College for Professional Studies for Teachers, Šabac Megatrend University, Beograd, Serbia
[b] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, Novi Sad 21000, Serbia

ARTICLE INFO

Article history:
Received 14 June 2013
Received in revised form 29 October 2013
Accepted 1 November 2013
Available online 23 November 2013

Keywords:
Fuzzy algebra
Fuzzy semigroup
Fuzzy identity
Fuzzy congruence
Fuzzy equality
Complete lattice

ABSTRACT

The notion of a fuzzy identity is introduced on a fuzzy subalgebra in the framework of fuzzy equalities instead of the crisp one. A fuzzy identity may be satisfied by a fuzzy subalgebra, while the underlying crisp algebra need not satisfy the analogous crisp identity. Among other properties, it is shown that for every fuzzy subalgebra μ of $\mathcal{A}$ there is a least fuzzy equality such that a fuzzy identity holds on μ. As an application, fuzzy semigroups are introduced with respect to a fuzzy equality. In particular, fuzzy semilattices, cancellative and regular fuzzy semigroups are investigated.

1. Introduction

We deal with the fuzzy approach to algebraic identities. Chronologically, fuzzy groups and semigroups, rings and other particular structures and applications (see [24,8,23,29,30]) were first defined and investigated. Investigations of notions from general algebra followed (see e.g., [11,26–28,4]). The universe of an algebra was fuzzified, while the operations remained crisp. The set of values was either the unit interval, or a complete, sometimes residuated lattice; generalized co-domains are also used (lattice ordered monoids, [21], posets or relational systems, [28]). An analysis of different co-domain lattices in the framework of fuzzy topology is presented by Höhle and Šostak in [15]. The notion of a fuzzy equality was introduced by Höhle [14] and then used by many others. Demirci (see e.g., [9,10]), considers particular algebraic structures equipped with a fuzzy equality relation; he also uses compatible fuzzy functions. Bělohlávek (the book [1], also [2,3] with Vychodil) introduces and investigates algebras with fuzzy equalities. These are defined as classical algebras in which the crisp equality is replaced by a fuzzy one being compatible with the fundamental operations of the algebra. Bělohlávek develops and investigates the most important fuzzified universal algebraic topics. Some aspects of universal algebra in a fuzzy framework were also investigated by Kuraoka and Suzuki, [19].

The main topic in the present paper is *fuzzy identities* on fuzzy subalgebras equipped with fuzzy equalities.

After Preliminaries, in Section 3 we introduce fuzzy congruences on fuzzy subalgebras of a crisp algebra. Fuzzy relations on fuzzy structures were introduced as a generalization of crisp relations on a subset of a set (see e.g., [13], and the book of Zimmermann, [35]). Fuzzy congruences are related to fuzzy subalgebras by a suitably defined reflexivity introduced several decades ago [34,13].

* Corresponding author. Tel.: +381 214852862; fax: +381 216350458.
E-mail addresses: budimirovic@sbb.rs (B. Budimirović), vjekoslav@sbb.rs (V. Budimirović), seselja@dmi.uns.ac.rs (B. Šešelja), andreja@dmi.uns.ac.rs (A. Tepavčević).

0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.11.007

In Section 4, we define our central notion, the fuzzy identities on a fuzzy subalgebra, as formulas in which terms are related by a fuzzy equality instead of the crisp one. A fuzzy identity may be satisfied by a fuzzy subalgebra (with respect to some fuzzy equality), while the underlying crisp algebra need not satisfy the analogous crisp identity. Our approach differs from the one by Bělohlávek and Vychodil [1–3]; in our case both, algebras and identities are fuzzy, implying other different consequences. We prove that *if a fuzzy subalgebra of an algebra satisfies a fuzzy identity with respect to some fuzzy equality, then there is the least fuzzy equality such that the corresponding fuzzy identity holds on the same fuzzy subalgebra.* We also prove that *a crisp algebra fulfils a crisp identity if and only if a particular fuzzy subalgebra satisfies the analogous fuzzy identity for all the corresponding fuzzy equalities.*

Section 5 deals with special fuzzy congruences on a crisp algebra, so called *weak* fuzzy congruences. We prove that *every weak fuzzy congruence on a crisp algebra is a fuzzy congruence on a particular fuzzy subalgebra; conversely, every fuzzy congruence on a fuzzy subalgebra of an algebra is a weak fuzzy congruence on this algebra.*

Section 6 contains the application in the semigroup theory. As it is known, semigroups provide a basic, algebraic tool in the analysis of regular languages and finite automata, also in the fuzzy framework. Among numerous papers and books we mention Pedrycz, Li, Lei, Malik, Mordeson, Kuroki [17,20–23,33]. We present concrete identities by which we introduce fuzzy semigroups on a fuzzy groupoid equipped with a fuzzy equality (therefore we call them *fuzzy E-semigroups*). We analyze basic related notions (idempotent, neutral and absorptive elements), we introduce fuzzy *E*-semilattices and prove that they can be equipped with an order (analogously to the crisp case). We also define fuzzy cancellative and regular *E*-semigroups, in which, as we show, it is possible to solve basic fuzzy equations.

Let us mention that our approach enables a new way of dealing with fuzzy automata and regular languages: fuzzy semigroups as introduced here fulfill the fuzzy associativity, they are defined on crisp groupoids which are not necessarily associative. Therefore, the recognisability in the fuzzy framework can be investigated using these new tools.

Another application of our results here is connected with regular *E*-semigroups: we are able to deal with problems of solving fuzzy equations, which turn out to be used in fuzzy control, discrete dynamic systems, knowledge engineering and other fields of information technology.

Our research partially originates in some ideas from our earlier paper [31], as well in our investigations in universal algebra, (papers [6,7,12,32]).

Due to a complete lattice as the membership values structure, our approach is cut-worthy: namely, crisp versions of generalized fuzzy properties hold on corresponding cut structures. The cut approach enables a research of fuzzy notions by investigating collections of their crisp substructures. Collections of cuts and their connection to fuzzy sets were analyzed by many authors, for recent results see papers of Jaballah and Saidi, e.g., [16,25].

2. Preliminaries

2.1. Algebras

We advance some notions from universal algebra, together with their relevant properties; more can be found e.g., in the book [5].

A **language** or a **type** $\mathcal{L}$ is a set $\mathcal{F}$ of functional symbols, together with a set of natural numbers (arities) associated to these symbols. An **algebra** of type $\mathcal{L}$ is a pair (A, F), denoted by $\mathcal{A}$, where A is a nonempty set and F is a set of (fundamental) operations on A. Each operation in F corresponds to some symbol in the language; if the symbol is n-ary, then the arity of the operation is n. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a non-empty subset of A. **Terms** in a language are usual regular expressions constructed by the variables and operational symbols (see [5] for the precise definition). If $v(x_1, \ldots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then $v^A : A^n \to A$ is the corresponding **term-operation** on $\mathcal{A}$. An **identity** in a language is a formula $t_1 = t_2$, where t_1, t_2 are terms in the same language. An equivalence relation ρ on A which is compatible with respect to all fundamental operations ($x_i \rho y_i, i = 1, \ldots, n$ imply $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$) is a **congruence** on $\mathcal{A}$.

2.2. Fuzzy sets, structures and relations

As the membership values structure we use a complete lattice, usually denoted by $(L, \wedge, \vee)$, with 0 and 1 being the smallest and the greatest element, respectively, under the order $\leqslant$ in L. Throughout the paper, a lattice is considered to be fixed and denoted by L, for short. Element x from a lattice L (or a poset P) is a **zero divisor** if there exists $y \neq 0$ such that $x \wedge y = 0$. Consequently, we say that a lattice L (or a subset P) is **zero divisor free** if for $x, y \in L$ (or $x, y \in P$), $x \wedge y = 0$ imply $x = 0$ or $y = 0$. A **semi-ideal** in an ordered set $(P, \leqslant)$ is its subset I fulfilling: for every $x \in P$ and $y \in I$, if $x \leqslant y$ then $x \in I$.

A **fuzzy set** μ on a nonempty set A is a function $\mu : A \to L$.

The **support** of $\mu : A \to L$ is a subset $\operatorname{supp}\mu$ of A, defined by

$$\operatorname{supp}\mu := \{x \in A | \mu(x) > 0\}.$$

For $p \in L$, a **cut set** or a p-**cut** of μ is a subset μ_p of X which is the inverse image of the principal filter in L, generated by p:

$$\mu_p = \{x \in X | \mu(x) \geqslant p\}.$$