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Using concept lattice theory to obtain the set of solutions of multi-adjoint relation equations ☆

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ABSTRACT

An important goal in the fuzzy relation equations framework is to obtain the whole set of solutions. This paper introduces algebraic properties in the multi-adjoint concept lattice setting. Moreover, using the relationship between these concept lattices and multi-adjoint relation equations, the complete set of solutions of these general equations is characterized and computed, avoiding the necessity of considering minimal solutions.

1. Introduction

Formal concept analysis (FCA), introduced by Wille [26], considers antitone Galois connections. On the other hand, rough set theory (RST) was proposed by Pawlak in the 1980s [20] and, later, Gediga and Düntsch [13] generalized this theory by presenting property-oriented concept lattices in order to consider two different sets following the philosophy of FCA (the set of objects and the set of attributes), which uses isotone Galois connections. Both, FCA and RST, are important formal tools for modeling and processing incomplete information in information systems.

Multi-adjoint formal concept lattices [19] and multi-adjoint property-oriented and object-oriented concept lattices [18] are fuzzy extensions of both theories, FCA and RST, respectively. These frameworks are flexible tools that could conveniently accommodate different fuzzy approaches presented in literature [1,5,14–16].

Fuzzy relation equations, introduced by Sanchez [24], are associated with the composition of fuzzy relations and have been used to investigate theoretical and applicational aspects of fuzzy set theory [8], e.g. approximate reasoning, time series forecast, decision making, fuzzy control, as an appropriate tool for handling and modeling of non-probabilistic forms of uncertainty, etc. Many papers have investigated the capacity to solve these equations.

Recently, the Galois connection theory has been successfully used to characterize solvability and find a set of solutions of systems of linear-like equations in semilinear spaces [21], which can be interpreted as fuzzy relation equations. In [10], the authors continued with these results and provided a narrow relationship between the solvability of a fuzzy relation equation and the theory of property-oriented concept lattices. These ideas and new results have been extended in a multi-adjoint setting in [9]. However, an important problem yet to be solved is finding and computing the complete set of solutions of a multi-adjoint relation equation.

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This paper focuses on both interconnected layers, multi-adjoint concept lattices and multi-adjoint relation equations. Firstly, we prove several interesting properties in the multi-adjoint property-oriented concept lattices, which can be developed in the other concept lattice frameworks, considering the ideas given in [2,18]. For example, a partition in the fuzzy subsets of objects and attributes is introduced, in which each equivalence class associated with the partition has only one concept; moreover, a procedure to obtain the subsets (equivalence classes) of this partition is given.

These properties provide fundamental consequences in the resolution of a multi-adjoint relation equation. In this paper, the complete set of solutions of a solvable multi-adjoint relation equation is characterized from the concepts of a specific concept lattice, without the necessity of considering the existence of minimal solutions as is assumed in other papers, such as in [17,22,25,27]. This characterization provides an interesting mechanism to compute the complete set of solutions. We also show that the current methods cannot be applied in the general multi-adjoint framework.

The plan of this paper is the following: in Section 2, the multi-adjoint property-oriented concept lattice framework is recalled and several new results about it are presented in Section 3. Later, Section 4 recalls multi-adjoint relation equations and characterizes the complete set of solutions from the previous properties. Lastly, the paper ends with several conclusions and prospects for future work.

2. Multi-adjoint property-oriented concept lattices

Multi-adjoint property-oriented and object-oriented concept lattices were introduced in [18] as a generalization of different fuzzy approaches of rough set theory [14,16].

The basic operators are the adjoint triples, which are formed by three mappings: a non-commutative conjunctor and two residuated implications, that satisfy the well-known adjoint property.

Definition 1. Let $(P_1,\leq_1)$, $(P_2,\leq_2)$, $(P_3,\leq_3)$ be posets and $\&: P_1\times P_2\to P_3$, $\swarrow: P_3\times P_2\to P_1$, $\nwarrow: P_3\times P_1\to P_2$ be mappings, then $(\&,\swarrow,\nwarrow)$ is called an *adjoint triple* with respect to $P_1,\ P_2,\ P_3$ if[1]:

$$x\leq_1 z\swarrow y \quad \text{iff} \quad x\& y\leq_3 z \quad \text{iff} \quad y\leq_2 z\nwarrow x$$

where $x\in P_1,\ y\in P_2$ and $z\in P_3$.

These operators are a straightforward generalization of a t-norm and its residuated implication. Since a t-norm is commutative, in this case both implications coincide. Several properties of these triples and the comparison with other operators can be found in [6], in which the following example is also given.

Example 1. Let $[0,1]_m$ be a regular partition of $[0,1]$ into m pieces, for example $[0,1]_2=\{0,0.5,1\}$ divides the unit interval into two pieces.

A discretization of a t-norm $\&: [0,1]\times[0,1]\to[0,1]$ is the operator $\&^*: [0,1]_n\times[0,1]_m\to[0,1]_k$, where $n,m,k\in\mathbb{N}$, and which is defined, for each $x\in[0,1]_n$ and $y\in[0,1]_m$, as:

$$x\&^* y=\frac{\lceil k\cdot(x\& y)\rceil}{k}$$

where $\lceil\_\rceil$ is the ceiling function.

For this operator, the corresponding residuated implications $\swarrow^*: [0,1]_k\times[0,1]_m\to[0,1]_n$ and $\nwarrow_*: [0,1]_k\times[0,1]_n\to[0,1]_m$ are defined as:

$$z\swarrow^* y=\frac{\lfloor n\cdot(z\leftarrow y)\rfloor}{n}\qquad z\nwarrow_* x=\frac{\lfloor m\cdot(z\leftarrow x)\rfloor}{m}$$

where $\lfloor\_\rfloor$ is the floor function and $\leftarrow$ is the residuated implication of the t-norm $\&$.

The triple $(\&^*,\swarrow^*,\nwarrow_*)$ is an adjoint triple, although the operator $\&^*$ could be neither commutative nor associative. □

The basic structure, which allows the existence of several adjoint triples, is the multi-adjoint property-oriented frame.

Definition 2. Given a poset $(P,\leqslant)$, two complete lattices $(L_1,\preceq_1)$ and $(L_2,\preceq_2)$ and adjoint triples with respect to $P,\ L_2,\ L_1,\ (\&_i,\swarrow^i,\nwarrow_i)$, for all $i\in\{1,\ldots,l\}$, the tuple $(L_1,L_2,P,\&_1,\ldots,\&_l)$ will be called *multi-adjoint property-oriented frame*.

The definition of context in this framework is analogous to the one given in [19].

Definition 3. Let $(L_1,L_2,P,\&_1,\ldots,\&_l)$ be a multi-adjoint property-oriented frame. A *context* is a tuple (A,B,R,σ), where A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R: A\times B\to P$ and $\sigma: B\to\{1,\ldots,l\}$ is a mapping which associates any element in B with a particular adjoint triple.

In order to make this contribution self-contained, we recall the formal definition of *isotone Galois connection*: Let $(P_1,\leq_1)$ and $(P_2,\leq_2)$ be posets, and $^\downarrow: P_1\to P_2$, $^\uparrow: P_2\to P_1$ mappings, the pair $(^\uparrow,^\downarrow)$ forms an *isotone Galois connection* between P_1 and P_2 if and only if: $^\uparrow$ and $^\downarrow$ are order-preserving; $x^{\downarrow\uparrow}\leq_1 x$, for all $x\in P_1$, and $y\leq_2 y^{\uparrow\downarrow}$, for all $y\in P_2$.

[1] Note that the antecedent will be evaluated on the right side, while the consequent will be evaluated on the left side, as in logic programming framework.