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Finite sets of data compatible with multidimensional inequality measures

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ABSTRACT

By using a general solution to the problem of extending a preorder conditional on a list of *ex-ante* comparisons between pairs, we ellucidate when a finite set of predetermined comparisons can be incorporated to a multidimensional inequality measure even if the population size is variable.

1. Introduction

The need for measures of multidimensional inequality arises when more than one criterion has to be applied in order to evaluate for example, welfare attached to different populations, household income inequality, or goodness for energy populations. A large number of analytical tools have been designed to compare distributions on this basis. Several proposals address the question from the viewpoint of attaching numerical values to each distribution (cf., e.g., [19] or [28] for general surveys). The most frequently used relative inequality index is the Gini index, a flexible tool for analysis that is also used e.g., as node splitting measure for decision tree construction (cf., [7] or [12] for recent references) or to derive welfare functions that can be used to compute energy welfare (cf., [17, Sec. 4.3]). But univariate inequality indices like the Gini index, the Theil index [23], or Atkinson's [5] indices do not give a full picture of the extent of inequality between groups of agents. This leads to constructions of multi-attribute inequality indices like the index developed by Maasoumi in [14–16]. It is constructed in two stages. In each step choices are made based on information theory: General Entropy measures are selected for both stages. Likewise, multi-attribute versions of e.g., Atkinson's index have been proposed in the literature (cf., [24]). Other methods for comparing pairs of distributions include dominance principles whose main handicap is incompleteness (cf., e.g., [6,18]).

The fact that a method of comparison in terms of inequality performs admittedly well is not incompatible with some degree of slackness in its prescriptions. For example: the U.S. Census Bureau historical table for measures of household income inequality in the country [25] claims that in the period 1988 to 1991, it increased according to the Gini measure but decreased according to the Theil measure. The same mixed evidence is observed in the period 1998 to 1999. The set of data that produced the historical table is the same. Nonetheless, if a researcher wants to use them to support the *ex-ante*

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prescription that household income inequality increased in the 1988–1991 and 1998–1999 lapses she can do it by means of a measure with good properties (namely, the Gini index), and similarly she can decide to support the opposite position (by appealing to the Theil index). In fact she can also use the same data to support the view that inequality increased in the 1988–1991 period but decreased in the 1998–1999 period by appealing to an orthodox procedure like Atkinson's measure with parameter 0.75.

Obviously, this slackness cannot be extended arbitrarily without violating desirable postulates. We are not aware of any analysis of the degree of slackness that is allowed when normative properties are imposed on complete methods of comparison. Our contribution intends to put forward this problem and present a first solution by referring to the recent approach by Savaglio [18], which has the remarkable feature that the assumption of fixed population size is dropped. Thus, in our proposal we first discuss properties that are desirable for a criteria in the current context. Then we consider a finite list of comparisons between distributions of goods or attributes to different populations (of possibly different sizes). Such list captures a given assessment of the inequality that those distributions convey to their respective populations. Finally, we check if this prescription is compatible with the existence of a criterion with the selected properties. In order to do so we take advantage of the recent Alcantud [2] ([8] for a broader discussion), where a general problem relating to extensions of preorders has been provided.

This paper is organized as follows. Section 2 gives some basic notation. Then in Section 3 we brief the reader on the conditional extension problem and its solution (Section 3.1), explain some specialized notation (Section 3.2), present some normative postulates (Section 3.3), describe the model (Section 3.4), and check for independence of its axiomatics (Section 3.5). Section 3.6 contains the solution to our problem directly from the arguments and results of Section 3.1. We summarize and address some related topics in Section 4.

2. Basic notation

Let $\mathbf{X}$ be a non-empty set. A binary relation R on $\mathbf{X}$ is a subset of $\mathbf{X}\times\mathbf{X}$. As is standard, $x\,R\,y$, is shorthand for $(x,y)\in R$. A reflexive and transitive relation is called a *preorder*, also called *quasiordering*. An *ordering* is a complete preorder.

The *asymmetric factor* P_R and the *symmetric factor* I_R of R are defined by

$$P_R=\{(x,y)\in X\times X|xRy \text{ and not } yRx\},$$
$$I_R=\{(x,y)\in X\times X|xRy \text{ and } yRx\}.$$

If R is a complete preorder then I_R is an equivalence relation. The shorthands P for P_R, $\widetilde{P}$ for $P_{\widetilde{R}}$, $\hat{P}$ for $P_{\hat{R}}, \ldots$ or I for I_R, $\widetilde{I}$ for $I_{\widetilde{R}}$, $\hat{I}$ for $I_{\hat{R}}, \ldots$ are common use.

If R and S are binary relations on $\mathbf{X}$ and $R\subseteq S$ then we say that R is *contained* or *included* in S. An *extension* of R binary relation on $\mathbf{X}$ is a binary relation S on $\mathbf{X}$ such that $R\subseteq S$ and $P_R\subseteq P_S$. Szpilrajn's Theorem [22], also [3, Th. 1.7] assures that every preorder can be extended to a complete preorder, i.e., it has an ordering extension.

3. An axiomatic approach to multidimensional inequality with initial constraints

In this Section we introduce the model that we intend to analyze and then solve our main question by means of recent advances in the theory of ordering extensions. Therefore we first brief the reader on the relevant approach to this technical problem in Section 3.1. Then we proceed to state and discuss the axioms under inspection and to set our model. After checking for independence of the postuates we prove the main result of the paper.

3.1. A fundamental result on conditional ordering extensions

Extending preorders to complete preorders has much appeal because it permits to work with relations that incorporate enough information as to apply significant tools of analysis, such as e.g., maximality results or utility assignments. Szpilrajn's theorem and its generalizations and variants are very often quoted and applied in many branches of mathematics and social sciences (for a very detailed description we refer to [4]). However, to the purpose of empirical contrast of the model we typically perform tests when only a finite amount of information is gathered. For that reason the applied researcher needs to be able to check when a finite number of exogenous comparisons can be matched with one such extension. The exact conditions under which we can extend a given preorder to a complete preorder conditional on a finite list of predetermined comparisons were given in Alcantud [2]. The following concept is the key to specify such solution.

Definition 1. Let $X_I=(a_1,\ldots,a_n,b_1,\ldots,b_n)$ be an ordered list of possibly repeated elements of $\mathbf{X}$, and R a preorder on $\mathbf{X}$. The R^A relation associated with X_I and R is given by $a_i\;R^A\;a_j$ if and only if $a_i\;R\;b_j$.

Remark 1. It is worth stating some particular cases of Definition 1. Under its assumptions, R^A is irreflexive if and only if a_iRb_i is false for each $i=1,\ldots,n$, because a_iRb_i amounts to $a_iR^Aa_i$. If $n=1$ then R^A is acyclic if and only if a_1Rb_1 is false. And if $n=2$ then R^A is acyclic if and only if the assertions a_1Rb_1, a_2Rb_2, and (a_1Rb_2 plus a_2Rb_1) are all false.