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The maximum flow problem of uncertain network ☆

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ABSTRACT

The maximum flow problem is one of the classic combinatorial optimization problems with many applications in electrical power systems, communication networks, computer networks and logistic networks. The goal of the problem is to find the maximum amount of flow from the source to the sink in a network. A network is called uncertain if the arc capacities of the network are uncertain variables. The main purpose of this paper is to solve the maximum flow in an uncertain network by under the framework of uncertainty theory.

1. Introduction

The maximum flow problem is one of the classic problems of network optimization. The motivation for the maximum flow problem came from the Soviet railway system [2]. As Ford and Fulkerson [12] explained, the maximum flow problem, formulated by Harris, is as follows: Consider a railway network connecting two cities by way of a number of intermediate cities, where each link of the network has an assigned number to represent its capacity. Assuming a steady state condition, we find a maximal flow from one given city to the other. Due to its many applications in electrical power systems, communication networks, computer networks and logistic networks, the maximum flow problem has been widely studied over the last 50 years. Therefore, from the viewpoint of application value, the study of maximum flow problem is of great significance.

The maximum flow problem has a long history. In the past five decades, many efficient algorithms for the maximum flow problem have emerged [36]. Representative methods in maximum flow algorithms are based on either augmenting paths or preflows [20,30]. Augmenting-path algorithms push flow along a path from the source to the sink in the residual network, and include Ford–Fulkerson's labeling algorithm [11,12] and Dinic's blocking flow algorithm [8]. Preflow-based algorithms push flow along edges in the residual network, and include Karzanov's blocking flow algorithm [29] and Goldberg–Tarjan's push-relabeling algorithm [38]. A class of graph algorithms has recently emerged, referred to as symbolic graph algorithms [9,17], based on which Gu and Xu presented the symbolic algorithms for the maximum flow in general networks [16]. In short, for a fixed network, the maximum flow by the above algorithms can be calculated when the arc capacities of the network are given. However, different types of uncertainty are frequently encountered in practice for a variety of reasons. For a

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new network, the arc capacities are uncertain in many situations. How the maximum flow of a network can be modeled and solved when in an uncertain environment is the main concern of this paper.

How do we consider the maximum flow of an uncertain network? Since under the uncertain environment the arc capacities of a network are uncertain, some researchers regarded the arc capacities as random variables [10,15,33] or fuzzy variables [3–5,18,19,37]. Such researchers employed probability theory or fuzzy theory to study the maximum flow problem. In other words, they mainly used stochastic optimization, chance constrained programming and robust optimization to solve the maximum flow problem in an uncertain network [1,31,32,34]. This paper will introduce the concept of the maximum flow function and apply uncertainty theory to the maximum flow problem in an uncertain network. This is a distinct contribution from other optimization methods. The similarities and differences between Liu's uncertainty concept and standard probabilistic concept, as well as other concepts of uncertainty could be found in [25].

In order to deal with some uncertain phenomena, uncertainty theory was proposed by Liu in 2007 and refined by Liu in 2010, and it has become a branch of mathematics for modeling human uncertainty. Uncertainty theory is different from probability theory and fuzzy theory. In uncertainty theory, uncertain measure is defined based on four axioms: normality axiom, duality axiom, subadditivity axiom and product axiom. Up to now, theory and practice have shown that uncertainty theory is an efficient tool to deal with nondeterministic information, especially expert data and subjective estimation. From a theoretical aspect, uncertain process [22,40], uncertain differential equation [6,41], and uncertain logic [27] have been established. From a practical aspect, uncertain programming [13,14,24,39], uncertain calculus [7,23], and uncertain risk analysis [26] have also developed quickly. In short, uncertainty theory is increasingly being researched and used. To explore the recent developments of uncertainty theory, readers may consult the book of Liu [25].

In this paper, we will illustrate that uncertainty theory can serve as a powerful tool to deal with the maximum flow in an uncertain network. The question is in what condition a network is suitable for application of the uncertainty theory? When we lack observed data for the arc capacities of a network, we often invite some domain experts to evaluate their belief degree that each event will occur, not only for economic reasons, but also for technical difficulties. However, since human beings usually tend to overweight unlikely events, the belief degree may have much larger variance than the real frequency, in which case probability theory is no longer valid. In this situation, uncertainty theory could be applied in network research.

2. Preliminaries

In this section, we will introduce some basic concepts and results of uncertainty theory.

Definition 1 (see [21]). Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ. A set function $\mathbb{M}:\mathcal{L}\to[0,1]$ is called an *uncertain measure* if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $\mathbb{M}\{\Gamma\}=1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathbb{M}\{\Lambda\}+\mathbb{M}\{\Lambda^c\}=1$ for any event $\Lambda\in\mathcal{L}$.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\ldots$, we have

$$\mathbb{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\leqslant\sum_{i=1}^{\infty}\mathbb{M}\{\Lambda_i\}.$$

The triple $(\Gamma,\mathcal{L},\mathbb{M})$ is called an *uncertainty space*. Besides, in order to provide an operational law, Liu defined the product uncertain measure on the product σ-algebra $\mathcal{L}$ as follows.

Axiom 4: (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathbb{M}_k)$ be uncertainty spaces for $k=1,2,\ldots$. Then the product uncertain measure $\mathbb{M}$ is an uncertain measure satisfying

$$\mathbb{M}\left\{\prod_{i=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathbb{M}\{\Lambda_k\}$$

where Λ_k are arbitrary chosen events from $\mathcal{L}_k$ for $k=1,2,\ldots$, respectively.

Definition 2 (see [21]). An uncertain variable is a measurable function ξ from an uncertainty space to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi\in B\}=\{\gamma\in\Gamma|\xi(\gamma)\in B\}$$

is an event.