Information Sciences 333 (2016) 81–87

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Diagonal plane sections of trivariate copulas

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ARTICLE INFO

Article history:
Received 17 July 2015
Revised 30 September 2015
Accepted 12 November 2015
Available online 27 November 2015

Keywords:
Aggregation function
Copula
Measure
Tail dependence

ABSTRACT

We introduce the notion of diagonal plane section of a trivariate copula as an additional tool to describe its tail dependence behavior. This notion extends the concept of diagonal section of a bivariate copula. We provide existence results for trivariate copulas with a given diagonal plane function. Related results are discussed, especially about extensions to a higher dimensional setting.

1. Introduction

The issue of aggregation of different sources of information into a single output has obtained a renovated interest in the last years that has also stimulated a novel treatment of its theoretical foundations. In particular, the formalization of the aggregation processes given by the so-called *aggregation functions* (see [15] for an overview) has underlined properties and relationships among various classes of such functions.

A copula is a special kind of n–ary aggregation function whose input values belong to $\mathbb{I} := [0, 1]$. Copulas are also Lipschitz continuous aggregation functions, i.e. the error in the aggregation output is linearly dependent on the error in the corresponding input terms (see [12,16] for more details). They are extensively used in statistics in order to aggregate one–dimensional probability laws into a joint (multivariate) probability law, keeping the marginal behavior fixed (see [18,23]). Such a feature is particularly helpful in building (flexible) high–dimensional distribution functions that take into account the available information about the lower–dimensional marginals and the ways they interact.

As stressed in [15], “*the choice of the aggregation function to be used is far from being arbitrary and should be based upon properties dictated by the framework in which the aggregation is performed*”. For example, in copula methods, the aim is often to calculate a risk measure associated with a multivariate model, and a possible relevant information that may constrained the choice is connected with the diagonal sections of its bivariate margins, since they describe conveniently the tail behavior of the associated random pairs (see, e.g., [6]). As such, various construction methods have been introduced assuming that the values of the copula are known along diagonals and other suitable sections. See, for instance, [3,4,10,17,19,25,26].

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http://dx.doi.org/10.1016/j.ins.2015.11.024
0020-0255/

In this paper, we aim at developing novel insights into this problem. Specifically, in Section 2, we present a possible extension of the concept of diagonal section of a bivariate copula to the three–dimensional case by introducing the so–called *diagonal plane section*. We discuss, hence, existence results for copulas with a given information on a specified subset of diagonal type (Section 3); furthermore, we show how they can be applied into a slightly different setting previously considered in [26] (Section 4). Section 5 is devoted to final comments about possible higher–dimensional generalizations of the introduced concepts.

2. Basic definitions and properties

Let $n \geq 2$ be an integer. An n-dimensional *copula* C (*n-copula*, for short) is an n-ary aggregation function with neutral element 1 that is n-increasing (see, e.g., [15]). Equivalently, C can be seen as a multivariate distribution function whose univariate margins are uniformly distributed on $\mathbb{I}$. Each n-copula C induces an n–fold stochastic measure μ_C on the Borel sets of $\mathbb{I}^n$ by setting $\mu_C(B) = V_C(B)$ for every n–rectangle $B \subseteq \mathbb{I}^n$, where $V_C(B)$ is the C–volume of B. Interestingly, the copula measure is also endowed in the construction of several generalized fuzzy integrals [22]. For more details about copulas, we refer to [12,23].

The *diagonal section* δ_C of an n–copula C is defined as the function given by $\delta_C(t) = C(t, \dots, t)$ for every $t \in \mathbb{I}$. It provides information about the tail concentration (see, e.g., [6]) of the copula C, i.e. the way the probability mass is distributed near the corners $(0, \dots, 0)$ and $(1, \dots, 1)$. In particular, δ_C can be used in order to calculate the tail dependence coefficients associated with C, when they exist (see, e.g., [13,18]).

If C is a 2–copula, then the diagonal section δ_C satisfies: (i) $\delta_C(1) = 1$, (ii) $\delta_C(t) \leq t$ for every $t \in \mathbb{I}$, and (iii) $0 \leq \delta_C(t') - \delta_C(t) \leq 2(t' - t)$ for every $t, t' \in \mathbb{I}$ such that $t \leq t'$. Any function $\delta : \mathbb{I} \to \mathbb{I}$ that satisfies (i), (ii) and (iii), is called a *diagonal function*.

The diagonal section contains the information about the values that an n–copula assumes on a set of (Hausdorff) dimension 1. Here, we provide a possible generalization of this concept when the values of the copula are known on a set of dimension 2. For the sake of simplicity, in the sequel we consider copulas in three dimensions.

Definition 2.1. The (1, 2)–*diagonal plane section* of a 3–copula C is the function $\Delta_C^{1,2} : \mathbb{I}^2 \to \mathbb{I}$ given by

$$\Delta_C^{1,2}(x, y) = C(x, x, y)$$

for every $(x, y) \in \mathbb{I}^2$. Similarly, the (1, 3)–*diagonal plane section* and the (2, 3)–*diagonal plane section* of C are defined by

$$\Delta_C^{1,3}(x, y) = C(x, y, x), \qquad \Delta_C^{2,3}(x, y) = C(y, x, x).$$

If (X, Y, Z) is a random vector (on a suitable probability space) distributed according to the copula C, then $\Delta_C^{1,2}$ represents the distribution function of $(\max(X, Y), Z)$. Similar statements apply to the other diagonal plane sections. As such, any diagonal plane section is a 2–increasing function. Obviously, any diagonal plane section contains the information about the diagonal section of C.

Example 2.1. Let D be the 2–copula that is obtained by spreading the probability mass uniformly on the segments joining (0, 1/2) with (1/2, 1), and (1/2, 1/2) with (1, 0). Consider the 3–copula $C(x, y, z) = zD(x, y)$. The diagonal plane sections associated with C are given by

$$\Delta_C^{1,2}(x, y) = yD(x, x), \quad \Delta_C^{1,3}(x, y) = xD(x, y), \quad \Delta_C^{2,3}(x, y) = xD(y, x).$$

Since D is not exchangeable, the diagonal plane sections are different. In fact, it is enough to consider that

$$\Delta_C^{1,2}(3/4, 1/4) = 1/8, \quad \Delta_C^{1,3}(3/4, 1/4) = 0, \quad \Delta_C^{2,3}(3/4, 1/4) = 3/16.$$

Notice that two bivariate margins of D coincide. Roughly speaking, the information given by the diagonal plane sections is not only related to the bivariate margins of a 3–copula, but also to the way they interact.

In the sequel, we concentrate on the (1, 2)–diagonal plane section, since the other cases can be obtained by a permutation of the arguments of the copula C. For simplicity, we set $\Delta_C := \Delta_C^{1,2}$.

As said, Δ_C has margins equal to the diagonal section of the copula $C_{12}(x, y) := C(x, y, 1)$ and the uniform distribution on $\mathbb{I}$, respectively. Specifically, the following properties are satisfied:

(D1) $\Delta_C(x, 1)$ is a diagonal function;
(D2) $\Delta_C(1, y) = y$ for every $y \in \mathbb{I}$;
(D3) Δ_C is 2–increasing.

Obviously, Δ_C is a 2–dimensional distribution function whose support is contained in $\mathbb{I}^2$. In particular, Sklar's theorem [12] ensures that, if $\delta_{C_{12}}$ denotes the diagonal section of C_{12},

$$\Delta_C(x, y) = D(\delta_{C_{12}}(x), y) \tag{1}$$

for all $(x, y) \in \mathbb{I}^2$ and a suitable 2–copula D.

If the random vector (X, Y, Z) is distributed according to C, Δ_C may also serve to describe the behavior of the conditional distribution function of $[X \leq x, Y \leq x | Z \leq y]$, which represents, roughly speaking, a kind of conditional version of the diagonal section of (X, Y).

Example 2.2. Let C be an Archimedean 3–copula (see, for instance, [1]) with additive generator φ. Then

$$\Delta_C(x, y) = \varphi^{-1}(2\varphi(x) + \varphi(y)).$$