Information Sciences 263 (2014) 36–42

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Weak convergence for random weighting estimation of smoothed quantile processes

Shesheng Gao [a], Yongmin Zhong [b,*], Chengfan Gu [c], Bijan Shirinzadeh [d]

[a] School of Automatics, Northwestern Polytechnical University, Xi'an 710072, China
[b] School of Aerospace, Mechanical and Manufacturing Engineering, RMIT University, Bundoora, VIC 3083, Australia
[c] School of Materials Science and Engineering, The University of New South Wales, Sydney, NSW 2052, Australia
[d] Robotics and Mechatronics Research Laboratory, Department of Mechanical and Aerospace Engineering, Monash University, Clayton, VIC 3800, Australia

ARTICLE INFO

Article history:
Received 17 November 2011
Received in revised form 21 September 2013
Accepted 22 November 2013
Available online 3 December 2013

Keywords:
Random weighting estimation
Smoothed quantile processes
Weak convergence

ABSTRACT

This paper presents a new random weighting method for smoothed quantile processes. A theory is established for random weighting estimation of smoothed quantile processes. It proves the weak convergence of the random weighting estimation error. Experiments and comparison analysis demonstrate that the proposed random weighting method can effectively estimate statistics, and the achieved accuracy and convergence speed are much higher than those of the Bootstrap method.

1. Introduction

Random weighting is a computational method in statistics. It has received great attention in the recent years, and has been widely applied to solve different problems [2,7–16,18,19,24,25]. However, there has been very limited research to use the random weighting method for estimation of quantile processes, which is an important research topic in the areas such as computational statistics and information technology [1,13,20,21]. Currently, quantile processes are commonly approximated by using the Bootstrap method [3–6,17,23]. In comparison with the Bootstrap method, the random weighting method has advantages [8,12,18,26]. It is simple in computation, suitable for large samples and unbiased in estimation, does not require the knowledge of the distribution function, and performs better than the Bootstrap method for small samples. Further, the statistical distribution generated by the random weighting method has a probability density function. This makes the random weighting method very suitable for computing a statistic required to have a probability density function. Recently, Gao et al. reported some preliminary results on random weighting estimation for quantile processes and associated confidence intervals [10,13]. However, these studies focused on non-smoothed quantile processes, rather than smoothed quantile processes.

This paper adopts the random weighting method for the first time to estimate smoothed quantile processes. It establishes a random weighting theory, showing that the stochastic behavior of the random weighting estimation error asymptotically converges to the behavior of the kernel density estimation for smoothed quantile processes. Experiments and comparison analysis have been conducted to comprehensively evaluate the performance of the proposed random weighting method for estimation of smoothed quantile processes.

* Corresponding author. Postal address: School of Aerospace, Mechanical and Manufacturing Engineering, RMIT University, PO Box 71, Bundoora, VIC 3083, Australia. Tel.: +61 3 99256018; fax: +61 3 99256108.

E-mail address: Yongmin.Zhong@rmit.edu.au (Y. Zhong).

0020-0255/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.ins.2013.11.031

2. Random weighting estimation of smoothed quantile processes

2.1. Random weighting method

Let $X_1, X_2, \ldots, X_n$ be a sample of independent and identically distributed random variables with common distribution function $F(x)$. Define the corresponding empirical distribution function of $F(x)$ as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(X_i \leqslant x)} \tag{1}$$

where $I_{(X_i \leqslant x)}$ is the indicator function.

Thus, the random weighting estimation of $F_n(x)$ can be written as

$$H_n(x) = \sum_{i=1}^{n} V_i I_{(X_i \leqslant x)} \tag{2}$$

where random vector $(V_1, V_2, \ldots, V_n)$ obeys Dirichlet distribution $D(1, 1, \ldots, 1)$, i.e. $\sum_{i=1}^{n} V_i = 1$ and the joint density function of $(V_1, V_2, \ldots, V_n)$ is $f(V_1, V_2, \ldots, V_n) = \Gamma(n)$, where Γ represents the Γ function, $(V_1, V_2, \ldots, V_{n-1}) \in S_{n-1}$ and $S_{n-1} = \left\{(V_1, V_2, \ldots, V_{n-1}) : V_i \geqslant 0,\ i = 1, \cdots, n-1, \sum_{i=1}^{n-1} V_i \leqslant 1\right\}$.

2.2. Theorem

Define the quantile q as

$$F^{-1}(q) = \inf\{x : F(x) \geqslant q\} \tag{3}$$

where $q \in (0,1)$.

$F^{-1}(q)$ can be estimated as

$$F_n^{-1}(q) = \inf\{x : F_n(x) \geqslant q\} \tag{4}$$

The objective is to approximate $P\left[\left(F_n^{-1}(q) - F^{-1}(q)\right) \leqslant t\right]$ by using the smoothed random weighting estimation $P^*\left[\left(\widehat{H}_n^{-1}(q) - \widehat{F}_n^{-1}(q)\right) \leqslant t\right]$, where $t \in R$, P^* is the conditional probability when $X_1, X_2, \ldots, X_n$ are given, and P represents the probability. $\widehat{H}_n^{-1}(q)$ and $\widehat{F}_n^{-1}(q)$ are defined as

$$\begin{aligned} \widehat{H}_n^{-1}(q) &= \inf\{t : \widehat{H}_n(t) \geqslant q\} \\ \widehat{F}_n^{-1}(q) &= \inf\{t : \widehat{F}_n(t) \geqslant q\} \end{aligned} \tag{5}$$

where $\widehat{H}_n(q)$ and $\widehat{F}_n(q)$ are the smoothed forms of $H_n(x)$ and $F_n(x)$.

The kernel estimation of $F(x)$ can be represented as

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{t - X_i}{\alpha_n}\right) = \int K\left(\frac{t - x}{\alpha_n}\right) F_n(x)dx \tag{6}$$

where $\alpha_n > 0$ is a series of positive constants such that $\alpha_n \to 0$ when $n \to \infty$, and the kernel function $K: R \to R$ is a distribution function.

Accordingly, the random weighting estimation of $\widehat{F}_n(x)$ is

$$\widehat{H}_n(t) = \sum_{i=1}^{n} V_i K\left(\frac{t - X_i}{\alpha_n}\right) = \int K\left(\frac{t - x}{\alpha_n} B_r\right) H_n(x)dx \tag{7}$$

where the integral is measured in the sense of Lebesgue, and B_r is an independent standard Brownian motion on $[0, \infty)$, resulting from the discrete form that obeys Brownian motion.

The kernel density estimation is defined as

$$\hat{f}_n(t) = \frac{1}{n\alpha_n}\sum_{i=1}^{n} k\left(\frac{t - X_i}{\alpha_n}\right) \tag{8}$$

where k is the density of kernel function K.

Theorem 1. *Suppose that*

(i) *F(x) is three-times continuously differentiable near* $F^{-1}(q)$;
(ii) $f = F'$ *and* $f(F^{-1}(q)) > 0$;