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On the topological properties of generalized rough sets

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ABSTRACT

In this paper, we consider some topological properties of generalized rough sets induced by binary relations and show that

1. Any serial binary relation can induce a topology.
2. Let R be a binary relation on a universe U. $t(R)$ and $e(R)$ denote the transitive closure and the equivalence closure of R, respectively. If R is a reflexive relation on U, then R and $t(R)$ induce the same topology, i.e. $T(R) = T(t(R))$. The interior and closure operators of the topology $T(R)$ induced by R are the lower and upper approximation operators $\underline{t(R)}$ and $\overline{t(R)}$, respectively. Moreover, $R(T(R)) = t(R)$, where $R(T(R))$ is the relation induced by the topology $T(R)$.
3. When R is a reflexive and symmetric relation, R and $e(R)$ induce the same topology, i.e. $T(R) = T(e(R))$. The interior and closure operators of the topology $T(R)$ induced by R are the lower and upper approximation operators $\underline{e(R)}$ and $\overline{e(R)}$, respectively. Moreover, $R(T(R)) = e(R)$.
4. Based on the above conclusions, the notion of topological reduction of incomplete information systems is proposed, and characterizations of reduction of consistent incomplete decision tables are obtained.

1. Introduction

Equivalence relation is one of the key notions in Pawlak rough set theory. Different kinds of generalizations of Pawlak rough set model can be obtained by replacing the equivalence relation with an arbitrary binary relation [11–13,20,25–28,31,33,34]. It is well-known that rough set theory is closely related to topology theory [1,3,4,7,16–19,27,29,32,35]. It was proved that there exists a one-to-one correspondence between the set of all reflexive and transitive relations and the set of all the Alexandrov topologies [18,29]. Many results about rough sets were obtained when the approximation space is finite. In the case where the universe is infinite, Michiro Kondo [3] studied the relationship between generalized rough sets induced by binary relations and topologies. He showed that $T(R) = \{X \subseteq U | \underline{R}(X) = X\}$ is a topology on U, when R is a reflexive relation. When R is a reflexive and symmetric relation, $T(R)$ is a topology such that X is open if and only if it is closed. However, when R is a reflexive relation, the lower approximation operator $\underline{R}$ is not an interior operator and the upper approximation operator $\overline{R}$ is not a closure operator. Two natural questions thus arise: (1) What are the relationships between the interior and closure operators of the topology $T(R)$ and the relation R? (2) What is the relationship between $R(T(R))$ and R, where $R(T(R))$ denotes the relation induced by the topology $T(R)$? In this paper we shall present some answers to these questions, and more importantly we do not need the finite assumption on the universe.

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This paper is organized as follows. In Section 2, we recall some main concepts and results about generalized rough sets. In Section 3, it is shown that a serial relation can induce a topology. In Section 4, we investigate the topological properties of generalized rough sets induced by a reflexive relation. In Section 5, we consider the case when R is a reflexive and symmetric relation. Some applications in incomplete information systems are given in Section 6. The paper ends with some conclusions in Section 7.

2. Generalized rough sets

Let U denote a non-empty set which we call the universe. Let R be an equivalence relation on U. The tuple (U,R) is called an approximation space. For any subset X of U, the lower and upper approximations of X are defined as follows:

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\},$$

where $[x]_R$ is the equivalence class of x with respect to R.

The Pawlak rough sets can be easily extended by considering other types of binary relations [2,20,23–28,31,33,34]. Let R be a binary relation on U without any additional constraints. $R_s(x) = \{y \in U | (x,y) \in R\}$ is called the successor neighborhood of x. When the relation is an equivalence relation, $R_s(x)$ is the equivalence class containing x. In general, by substituting $[x]_R$ with $R_s(x)$, we have the following definition of generalized rough sets.

Definition 2.1. [14,26–28,31]. Let U be a universe, and R a binary relation on U. The tuple (U,R) is called a generalized approximation space. For any set $X \subseteq U$, the lower approximation $\underline{R}(X)$ and the upper approximation $\overline{R}(X)$ of X are, respectively, defined as

$$\underline{R}(X) = \{x \in U | R_s(x) \subseteq X\},$$
$$\overline{R}(X) = \{x \in U | R_s(x) \cap X \neq \emptyset\}.$$

Furthermore, if $\underline{R}(X) = \overline{R}(X)$, then X is said to be definable.

In a generalized approximation space, one may define approximation operators in many different ways. For example, one can define another definition of approximation operators. For any set $X \subseteq U$, a pair of lower and upper approximations, $\underline{apr}(X)$ and $\overline{apr}(X)$, are defined by:

$$\underline{apr}(X) = \bigcup\{R_s(x) | R_s(x) \subseteq X\},$$
$$\overline{apr}(X) = \bigcup\{R_s(x) | R_s(x) \cap X \neq \emptyset\}.$$

The above definition is not equivalent to Definition 2.1. The approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. But the approximation operators $\underline{apr}$ and $\overline{apr}$ are not necessarily dual operators. The duality property of operators is very important in topological spaces. Therefore, we will focus on the approximation operators in Definition 2.1 in the sequel.

The following theorem gives main properties of two approximation operators in Definition 2.1.

Theorem 2.2. [26,28,31]. *In a generalized approximation space (U,R), the operators $\underline{R}$ and $\overline{R}$ satisfy the following properties: For any $X, Y \subseteq U$*

(1) $\underline{R}(X) = \sim \overline{R}(\sim X), \overline{R}(X) = \sim \underline{R}(\sim X)$, *where* $\sim X = U - X$.
(2) $\underline{R}(U) = U, \overline{R}(\emptyset) = \emptyset$.
(3) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y), \overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$.
(4) *If* $X \subseteq Y$, *then* $\underline{R}(X) \subseteq \underline{R}(Y), \overline{R}(X) \subseteq \overline{R}(Y)$.
(5) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y), \overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$.

A binary relation R on U is said to be serial if for all $x \in U$, there exists $y \in U$ such that $(x,y) \in R$; R is said to be reflexive if for all $x \in U$, we have $(x,x) \in R$; R is said to be symmetric if for all $x, y \in U$, we have $(x,y) \in R$ implies $(y,x) \in R$; R is said to be transitive if for all $x, y, z \in U$, we have $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$. An equivalence relation on U is a reflexive, symmetric and transitive relation on U.

The following results hold for the rough set models induced by several different kinds of binary relations.

Theorem 2.3. [26,28,31]. *In a generalized approximation space (U,R), we have*

(i) *R is serial if and only if one of the following three conditions holds for any $X \subseteq U$:*
(1) $\underline{R}(X) \subseteq \overline{R}(X)$, (2) $\underline{R}(\emptyset) = \emptyset$, (3) $\overline{R}(U) = U$.
(ii) *R is reflexive if and only if one of the following two conditions holds for any $X \subseteq U$:*
(1) $\underline{R}(X) \subseteq X$, (2) $X \subseteq \overline{R}(X)$.