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Some sets of indistinguishability operators as multiresolution families

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ABSTRACT

Multiresolution is a general mathematical concept that allows us to study a property by means of several changes of resolution. From a fixed resolution, a coarser projection can be calculated and then the changes between a finer resolution and a coarser one can be studied. That information can give a good knowledge about the problem under consideration. Also using multiresolution techniques it is possible to present information with a higher or a lower detail, given a way to get the adequate granularity or abstraction for a context.

The granularity of a system can be obtained or modeled by the use of indistinguishability operators. In this work the relation between indistinguishability operators and multiresolution theory is studied and several methods to build families of indistinguishability operators with multiresolution capacities are given.

1. Introduction

Multiresolution Analysis [21,22] is the basic mathematical tool to build wavelet systems in Wavelet Theory [4,5,24,15,30] and it has broadly been used in image processing by applying it in compression, feature selection, denoising and other image processing tasks, besides of other multiply applications [34]. However the multiresolution concept is a more general and a very interesting concept and it can be generalized and applied to multiple fields of data processing [11]. It also can be considered as an example of a Hierarchical System [29,25].

In its basic and common functional spaces formulation, a multiresolution system consists of a family of nested functional subspaces that is dense in the whole functional space and has empty intersection. The projections of a function into each subspace give an approximation to this function with different accuracy. Two functions can have the same projection into a subspace but different projection into another one. So it can be possible that two different functions can be indistinguishable into a subspace but distinguishable in another one. An approximation to a function in a functional subspace entails a certain form of making indistinguishable some kind of functions: the approximation considers equal all functions with differences smaller than its accuracy level. In this sense, each level in a multiresolution analysis implicitly has associated an indistinguishability operator on the elements of the functional space.

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A multiresolution analysis requires some other, and perhaps more important, properties. In the first place, it must be possible to go from one level to another by means of dyadic expansions or contractions. In the second place, there must exist a function, named scaling function, that generates by integer shifts and dilations a basis of the functional subspaces. There are several such functions, especially B-spline functions [6], that can be used as fuzzy sets and at the same time generating a multiresolution analysis.

A Multiresolution Analysis induces in a natural way a family of distances in the functional space: it is enough to take the Euclidean distance in each functional subspace. This family of distances inherits some multiresolution characteristics.

In Fuzzy Set Theory the concept of indistinguishability operator is a generalization of the equivalence relation concept in the crisp field and includes the concept of similarity relation [37]. They have been widely studied and there is a good knowledge of their structure and properties [33,32,28,3]. Indistinguishability operators allow us to measure the degree of similarity or indistinguishability between elements of a universe of discourse in a coherent way. Different indistinguishability operators will then generate different granularity on the universe of discourse and in this way are good tools to generate a Multiresolution analysis. In this work the possibility to define families of indistinguishability operators with multiresolution properties is studied. Going through the different members of one of such families will allow us to refine or coarsen the granularity of the system as required for a concrete purpose.

2. A basic scheme for a multiresolution analysis

Basically a multiresolution system (MR) allows us to fix an observation system over a universe X. If the MR system is good enough we will be able to observe with any level of detail the property of the elements of X that we are considering. But at the same time we can delete, if we need, any difference between elements of X, as far as that property is considered, because, for example, we are not interested in those differences. In fact the changes in the resolution level must give us enough information about the characteristics of the elements which we are interested in.

In [12] a very general framework for a scheme of a multiresolution representation of data is given. Basically a basic scheme of multiresolution can be defined as follows.

Definition 2.1. A basic scheme of multiresolution is a triplet $(\mathcal{F}, (V_k)_{k=0}^{\infty}, \{\mathcal{D}_{k=0}^{\infty}\})$ where $\mathcal{F}$ is a vector space, $(V_k)_{k=0}^{\infty}$ a sequence of vector spaces and $\{\mathcal{D}_{k=0}^{\infty}\}$ a sequence of linear operators $D_k : \mathcal{F} \longrightarrow V_k$ that satisfies two properties:

1. D_k are onto mappings,
2. $D_k(f) = 0 \Rightarrow D_{k-1}(f) = 0$.

Using the family of operators D_k two sets of operators can be defined:

1. a set of decimation operators $D_k^{k-1} : V_k \longrightarrow V_{k-1}$ that allows us going from a greater resolution level to a lower resolution level,
2. and a set of prediction operators $P_{k-1}^k : V_{k-1} \longrightarrow V_k$ that allows us, using suitable data, going from a lower resolution level to a greater resolution level.

The operators D_k^{k-1} can be defined by means of the expression $D_k^{k-1}(v_k) = D_{k-1}(f)$ where f is any element of $\mathcal{F}$ such as $D_k(f) = v_k$. These operators are well defined thanks to the second property of Definition 2.1.

The family of operators P_{k-1}^k are built thanks to the first property of Definition 2.1. As the operators D_k are onto, there must exists at least a right inverse $R_k : V_k \longrightarrow \mathcal{F}$ for each one. Taking a family $\{R_k\}$ of them it is possible to define the operators P_{k-1}^k as $P_{k-1}^k = D_k \cdot R_{k-1}$. Under these conditions it is easy to prove that P_{k-1}^k is a right-inverse of the decimation operator D_{k-1}^k but not a left-inverse (see [12] for details).

Given an element $f \in \mathcal{F}$ it is possible to take their projection at a fixed level k. Let $v_k = D_k(f)$ be this projection. Using the decimation operators we can get a sequence of values as $v_{i-1} = D_i^{i-1}(v_i)$. Each value v_i is a coarser representation of the original value f. On the other hand from each value v_{i-1} it is possible to predict the value for the level i from the value of level $i-1$ by means of $v_i' = P_{i-1}^i(v_{i-1})$.

The relation between the decimation operators and the prediction operators is essential to get a multiresolution system. These operators must have a certain type of invertible relation between them but it must not be perfect, because if it were perfect, we would have the same information in each level. Usually the prediction operators need some additional information to be able to reconstruct the original projection. This extra information is called the detail information.

In the frame of vectorial spaces the detail coefficients are defined as the difference between the values v_k and v_k', i.e. $d_k = v_k - v_k'$. In this frame the detail coefficients belong to the kernel of the decimation operators. Thanks to the existence of vector basis in the vector spaces V_k it is possible to write the expressions of the elements v_k, v_k' and d_k. Then there is a one-to-one correspondence between v_k and $\{v_{k-1}, d_k\}$ because the decimation and prediction operators allows us to obtain $\{v_{k-1}, d_k\}$ from v_k and the prediction operator allows us to obtain again v_k from $\{v_{k-1}, d_k\}$. The process can be iterated for $k = N \dots 1$ obtaining a one-to-one mapping from v_N to $d_N, d_{N-1}, \dots, d_1, v_0$. Analyzing the values $d_N, d_{N-1}, \dots, d_1$ it is possible