Information Sciences 305 (2015) 1–13

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

# Matchings extend to Hamiltonian cycles in $k$-ary $n$-cubes ☆

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ARTICLE INFO

*Article history:*
Received 20 January 2014
Received in revised form 1 November 2014
Accepted 1 February 2015
Available online 7 February 2015

*Keywords:*
Interconnection network
$k$-ary $n$-cube
Hamiltonian cycle
Perfect matching
Matching

ABSTRACT

The $k$-ary $n$-cube is one of the most popular interconnection networks for parallel and distributed systems. Given an edge set in the $k$-ary $n$-cube, which conditions guarantee the existence of a Hamiltonian cycle in the $k$-ary $n$-cube containing the edge set? In this paper, we prove for $n \geqslant 2$ and $k \geqslant 3$ that every matching having at most $3n-8$ edges is contained in a Hamiltonian cycle in the $k$-ary $n$-cube. Also, we present an example to show that the analogous conclusion does not hold for perfect matchings.



## 1. Introduction

The $k$-ary $n$-cube is one of the most popular interconnection networks for parallel and distributed systems; see [5,8–10]. There is a large amount of literature on topological properties of $k$-ary $n$-cube, such as ease of implementation, low-latency and high-bandwidth interprocessor communication [5,9,15]. Also, it has drawn considerable research attention to its applications in parallel computing [20], online analytical processing of data [28], cloud computing [17], analysis of multidimensional text databases [30], etc. Several parallel systems, such as iWarp [4,21], Cray T3D [18], Cray T3E [2] and the IBM Blue Gene [1], have been built based on the $k$-ary $n$-cube.

The *$k$-ary $n$-cube*, denoted by $Q_n^k$, where $n \geqslant 1$ and $k \geqslant 2$, is a graph consisting of $k^n$ vertices, each of which has the form $u = \delta_1 \cdots \delta_n$, where $0 \leqslant \delta_i \leqslant k-1$ for every $i \geqslant 1$. Two vertices $u = \delta_1 \cdots \delta_n$ and $v = \beta_1 \cdots \beta_n$ are adjacent if and only if there exists an integer $j \in \{1, \ldots, n\}$ such that $\delta_j = \beta_j \pm 1 \pmod k$ and $\delta_i = \beta_i$ for every $i \in \{1, \ldots, n\} \setminus \{j\}$. The $Q_n^2$ is the well-studied hypercube. The $Q_1^k$ with $k \geqslant 3$ is a cycle of length $k$.

Since some parallel applications such as those in image and signal processing are originally designated for a path or cycle architecture, it is important to investigate cycle and path embeddings in an interconnection network [3,6,7,11–13,25–27,29].

It is well known that $Q_n^k$ is Hamiltonian [5]. Given an edge set in the $k$-ary $n$-cube, which conditions guarantee the existence of a Hamiltonian cycle in the $k$-ary $n$-cube containing the edge set? A forest is *linear* if each component of it is a path. Caha and Koubek [6] investigated the Hamiltonian cycle and path embedding problem in the 2-ary $n$-cube with prescribed edges, and they observed that any proper subgraph of a Hamiltonian cycle must be a linear forest. Dvořák [11] showed for $n \geqslant 2$ that every linear forest containing at most $2n-3$ edges of an 2-ary $n$-cube can be embedded into a Hamiltonian cycle.

☆ This work is supported by NSFC (Grant Nos. 61073046 and 11371180).
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http://dx.doi.org/10.1016/j.ins.2015.02.005
0020-0255/

Wang et al. [27] and Stewart [23] investigated the case when $k \geqslant 3$, and obtained the following result, respectively.

**Theorem 1.1** *([23,27]). For $n \geqslant 2$ and $k \geqslant 3$, if $F$ is a linear forest in $Q_n^k$ with $|F| \leqslant 2n-1$, then $F$ is contained in a Hamiltonian cycle in $Q_n^k$.*

A set of edges in a graph $G$ is called a *matching* if no two edges have an end-vertex in common. A matching in $G$ is *perfect* if it covers all vertices of $G$. On the other hand, Ruskey and Savage [22] asked the following question: For $n \geqslant 2$, does every matching in a hypercube $Q_n$ extend to a Hamiltonian cycle in $Q_n$? Kreweras [19] conjectured for $n \geqslant 2$ that every perfect matching in $Q_n$ extends to a Hamiltonian cycle in $Q_n$. Fink [13,14] confirmed the conjecture to be true.

However, the analogous result does not necessarily hold for $Q_n^k$. For example, a perfect matching $M$ in $Q_2^6$ as shown in Fig. 1 cannot be contained in any Hamiltonian cycle in $Q_2^6$. We can verify this by a computer program: start with the vertex 00, and search for an $M$-alternating Hamiltonian cycle in $Q_2^6$ by examining all possibilities of $M$-alternating paths and cycles.

In this paper, we consider the problem of embedding Hamiltonian cycles in the $k$-ary $n$-cubes with small matchings, and obtain the following main result.

**Theorem 1.2.** *For $n \geqslant 2$ and $k \geqslant 3$, if $M$ is a matching in $Q_n^k$ with $|M| \leqslant 3n-8$, then $M$ is contained in a Hamiltonian cycle in $Q_n^k$.*

The rest of the paper is organized as follows. In Section 3 we present the proof of the main result. Section 2 presents some results about spanning $k$-paths which are applied in the proof of the main result.

## 2. Spanning $k$-paths in $Q_n^k$

This section is devoted to auxiliary results about spanning $k$-paths which are applied in the constructions of Hamiltonian cycles in the main result. First, we introduce two classical results about Hamiltonian paths. After that we present some generalizations of the two results. Since some results are used in the proof of the main theorem frequently, we formulate the repeating operations as lemmas. Next, we present some results about spanning 2-paths.

Before that, we introduce some necessary definitions. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For an edge $e$, we use $V(e)$ to denote the set of the two endpoints of $e$. For a set $F \subseteq E(G)$, let $V(F) = \bigcup_{e \in F} V(e)$.

For a set $U \subseteq V(G)$, let $G - U$ denote the resulting graph by deleting from $G$ the vertices in $U$ together with all the edges incident with these vertices. For a set $F \subseteq E(G)$, let $G - F$ denote the graph with the vertex set $V(G)$ and edge set $E(G) \setminus F$.

Let $H$ and $H'$ be two subgraphs of $G$. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. Let $F \subseteq E(G) \setminus E(H)$. We use $H + F$ to denote the graph with the vertex set $V(H) \cup V(F)$ and edge set $E(H) \cup F$. When $F = \{e\}$, we simply write $H + F$ as $H + e$.

The *distance* between vertices $u$ and $v$ in $G$ is the number of edges in a shortest path joining $u$ and $v$, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear. The distance of a vertex $u$ and an edge $xy$ is defined by $d(u, xy) = \min\{d(u, x), d(u, y)\}$.

A *$u, v$-path* is a path with endpoints $u$ and $v$, denoted by $P_{u,v}$ when we specify a particular such path. In a graph $G$, a spanning subgraph without isolate vertices is called a *spanning $k$-path* if it consists of $k$ disjoint paths. A spanning 1-path thus is simply a spanning or Hamiltonian path.

In the following, we always assume $n \geqslant 2$ and $k \geqslant 3$. Each vertex in $Q_n^k$ has $2n$ neighbors in $Q_n^k$, and $Q_n^k$ is bipartite if and only if $k$ is even. When $k$ is even, the *parity* of a vertex $u = \delta_1 \cdots \delta_n$ in $Q_n^k$ is defined by $p(u) = \sum_{i=1}^{n} \delta_i \pmod 2$, and thus the condition "$p(u) \neq p(v)$" is necessary for the existence of a spanning $u, v$-path in $Q_n^k$ with even $k$. Note that for even $k$, if $uv \in E(Q_n^k)$, then $p(u) \neq p(v)$.

First let us recall the following classical results.

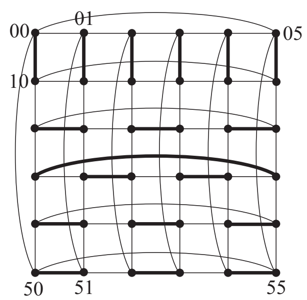


**Fig. 1.** Perfect matching $M$ which is not contained in any Hamiltonian cycle in $Q_2^6$.