Information Sciences 250 (2013) 178–183

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

The relationship among different covering approximations

Guilong Liu *

School of Information Science, Beijing Language and Culture University, Beijing 100083, China

ARTICLE INFO

Article history:
Received 8 April 2011
Received in revised form 10 July 2013
Accepted 27 July 2013
Available online 2 August 2013

Keywords:
Binary relation
Covering
Approximation
Rough set
Neighborhood

ABSTRACT

This paper studies covering-based rough sets by means of characteristic functions of sets. We consider covering-based rough sets from both constructive and axiomatic approaches. The relationship among four types of covering-based rough sets is discussed. We also outline the topologies induced by different covering approximations.

1. Introduction

Pawlak's rough set theory [9–11] is based on equivalence relations. However, the requirement of the equivalence relations appears to be a very restrictive condition that may limit the application of rough sets. To address this issue, many authors have generalized the concept of rough sets to arbitrary binary relations [3,15,19,20,23], fuzzy relations [1,2], and coverings [6,14,22,25,26]. Zakowski [22] first established covering-based rough set theory by exploiting coverings of a universal set. Covering-based rough set theory, which is an important generalization of classical rough set theory, is also a model with promising potential for application to data analysis. Zhu and Wang [26] studied many types of covering-based rough sets, while Qin et al. [12] proposed five pairs of dual covering approximations using the notion of a neighborhood. Yun et al. [21] also studied covering-based rough sets and solved an open problem identified by Zhu and Wang [26].

The motivation of the paper is to consider the following problem: What is the relationship among covering approximations $C_i(i = 3–6)$? This paper considers the covering approximations proposed by Qin et al. [12]. Using a constructive approach, we obtain the relationship between approximations C_3, C_4, C_5, and C_6. We show that $\overline{C_4}$ is a composition of $\underline{C_3}$ and $\overline{C_3}$ and that $\overline{C_5}$ is a composition of $\overline{C_6}$ and $\overline{C_3}$. We also give characterizations of covering upper approximations $\overline{C_i}(1 = 2-6)$. We further investigate the conditions of neighborhoods, $\{N(x)|x \in U\}$, forming a partition of U. This paper simplifies the study of covering approximations, C_i $(i = 3–6)$.

The paper is arranged as follows. Section 2 reviews the main ideas of generalized rough set and covering approximations. Section 3 gives the properties of approximations $C_i(i = 1–6)$. Section 4 studies axiomatization of covering upper approximations $C_i(i = 1–6)$, while Section 5 considers the conditions of the neighborhoods, $\{N(x)|x \in U\}$, forming a partition of U. Section 6 discusses topologies induced by $C_i(i = 3–6)$. Finally, Section 7 concludes the paper.

* Tel.: +86 010 82387845.
E-mail address: liuguilong@blcu.edu.cn

0020-0255/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.ins.2013.07.019

2. Generalized rough sets and covering approximations

Let U be a non-empty set of objects called the universal set and $P(U)$ be the power set of U. Suppose that R is an arbitrary relation on U. Recall that the left and right R-relative sets of an element x in U are defined as

$$l_R(x) = \{y|y \in U, yRx\} \text{and} r_R(x) = \{y|y \in U, xRy\},$$

respectively. By replacing equivalence classes with right R-relative sets, Yao [18] defined the operators $\underline{R}$ and $\overline{R}$ from $P(U)$ to itself as

$$\underline{R}(X) = \{x|r_R(x) \subseteq X\} \text{ and } \overline{R}(X) = \{x|r_R(x) \cap X \neq \emptyset\}.$$

$\underline{R\,X}$ is called the lower approximation of X and $\overline{R}X$ the upper approximation of X. The pair $(\underline{R}X, \overline{R}X)$ is referred to as a generalized rough set based on R.

Recall that if X is a subset of universal set U, the characteristic function λ_X of X is defined for each $x \in U$ as follows: $\lambda_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases}$. Similarly, for any given binary relation R on U, for $x,y \in U$, $\lambda_R(x,y)$ is equal to 1 if xRy and to 0 otherwise. The inner product of two subsets X,Y of U, denoted as (X,Y), is defined as $(X,Y) = \vee_{x \in U}(\lambda_X(x) \wedge \lambda_Y(x))$, where $\wedge$ denotes the minimum and $\vee$ the maximum.

Let C denote a covering of U. We call the ordered pair (U,C) a covering approximation space and $N(x) = \cap\{K|K \in C, x \in K\}$ the neighborhood of an element $x \in U$. Throughout this paper, for any given covering approximation space (U,C), the relation R on U is defined as $l_R(x) = N(x)$ for each $x \in U$. It is easily verified that R is reflexive and transitive. This paper considers six types of covering approximation operations defined as follows [12,14,21]:

(1) $\underline{C_1}(X) = \cup\{K|K \in C, K \subseteq X\}$ and $\overline{C_1}(X) = \cup\{K|K \in C, K \cap X \neq \emptyset\}$.
(2) $\underline{C_2}(X) = \cup\{K|K \in C, K \subseteq X\}$ and $\overline{C_2}(X) = (\underline{C_2}(X'))'$, where X' denotes the complement of X in U.
(3) $\underline{C_3}(X) = \{x|N(x) \subseteq X\}$ and $\overline{C_3}(X) = \{x|N(x) \cap X \neq \emptyset\}$.
(4) $\underline{C_4}(X) = \{x|\exists u \in N(x), N(u) \subseteq X\}$ and $\overline{C_4}(X) = \{x|\forall u(u \in N(x) \to N(u) \cap X \neq \emptyset)\}$.
(5) $\underline{C_5}(X) = \{x|\forall u(x \in N(u) \to N(u) \subseteq X)\}$ and $\overline{C_5}(X) = \cup\{N(x)|N(x) \cap X \neq \emptyset\}$.
(6) $\underline{C_6}(X) = \{x|\forall u(x \in N(u) \to u \in X)\}$ and $\overline{C_6}(X) = \cup_{x \in X} N(x)$.

3. Properties of covering approximations

Qin et al. [12] proposed five pairs of dual approximations, with C_3, C_4, C_5, and C_6 defined by means of neighborhood $N(x)$. It is natural to ask the question: Is there a close relationship among C_3, C_4, C_5, and C_6? The following theorem answers this question.

Theorem 3.1. *Let (U,C) be a given covering approximation space.*

(1) *If we define a binary relation S on U through $l_S(x) = \cup\{K|K \in C, x \in K\}$ for each $x \in U$, then S is reflexive and symmetric.*
(2) $\overline{C_1}(X) = \overline{S}X$ *for each subset* $X \subseteq U$ [8].
(3) $\overline{C_2}X = \cap\{K'|K \in C, X \subseteq K'\}$.

(4) *$\{N(x)|x \in U\}$ forms a partition of U if and only if R is symmetric.*
(5) $\underline{C_3}(X) = \underline{R^{-1}}X$ *and* $\overline{C_3}(X) = \overline{R^{-1}}X$ *for each $X \subseteq U$, where R^{-1} denotes the inverse of R.*
(6) $\underline{C_4}(X) = R^{-1}(\underline{R^{-1}}(X))$ and $\overline{C_4}(X) = \underline{R^{-1}}(\overline{R^{-1}}(X))$ *for each $X \in P(U)$.*
(7) $\underline{C_5}(X) = \underline{R}(\underline{R^{-1}}(X))$ *and* $\overline{C_5}(X) = \overline{R}(\overline{R^{-1}}(X))$.
(8) $\underline{C_6}(X) = \underline{R\,X}$ *and* $\overline{C_6}(X) = \overline{R}X$ *for each* $X \subseteq U$ [8].

Proof

(4) By the definition of relation R, $\{N(x)|x \in U\}$ forms a partition of U if and only if R is an equivalence relation, implying that R is symmetric.
(6) $\overline{C_4}(X) = \{x|\forall u(u \in N(x) \to N(u) \cap X \neq \emptyset)\} = \{x|\forall u(u \in l_R(x), \to l_R(u) \cap X \neq \emptyset)\} = \{x|\forall u \in l_R(x), u \in \overline{R^{-1}}(X)\} = \{x|l_R(x) \subseteq \overline{R^{-1}}(X)\} = \underline{R^{-1}}(\overline{R^{-1}}(X))$. By duality, $\underline{C_4}(X) = (\underline{R^{-1}}(\overline{R^{-1}}(X')))' = \overline{R^{-1}}(\underline{R^{-1}}(X))$.
(7) Because $\overline{C_5}\{y\} = \cup\{N(x)|y \in N(x)\} = \cup\{l_R(x)|y \in l_R(x)\} = \cup\{l_R(x)|yRx\} = \cup_{x \in r_R(y)} l_R(x) = \overline{R}r_R(y) = \overline{R}\overline{R^{-1}}(\{y\})$ for each $y \in U$, we have $\overline{C_5}(X) = \cup_{y \in X}\overline{C_5}\{y\} = \cup_{y \in X}\overline{R}(\overline{R^{-1}}(\{y\})) = \overline{R}(\overline{R^{-1}}(X))$ for each $X \in P(U)$. By duality, $\underline{C_5}(X) = \underline{R}(\underline{R^{-1}}(X))$. We omit the proof of the remaining parts because these can be proven in a similar way. □

Theorem 3.1 tells us that $\overline{C_4}$ is a composition of $\underline{C_3}$ and $\overline{C_3}$, and $\overline{C_5}$ is a composition of $\overline{C_6}$ and $\overline{C_3}$.